NUWC-NPT Technical Report 10,760
5 May 1997

Performance of Power-Law Processor with Normalization for Random Signals of Unknown Structure

Albert H. Nuttall
Surface Undersea Warfare Directorate

Naval Undersea Warfare Center Division
Newport, Rhode Island

PREFACE

The work described in this report was sponsored by the Independent Research (IR) Program of the Naval Undersea Warfare Center (NUWC), Division Newport, under Project No. B100077, "Near-Optimum Detection of Random Signals with Unknown Locations, Structure, Extent, and Strengths," principal investigator Albert H. Nuttall (Code 311). The IR program is funded by the Office of Naval Research; the NUWC Division Newport program manager is Stuart C. Dickinson (Code 102). This research was also sponsored by the Science and Technology Directorate of the Office of Naval Research, Ronald Tipper (ONR-322B).

The technical reviewer for this report was Stephen G. Greineder (Code 2121).

Reviewed and Approved: 5 May 1997

Patricia J. Dean

Patricia J. Dean
Director, Surface Undersea Warfare

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	5 May 1997	Progress

4. TITLE AND SUBTITLE
Performance of Power-Law Processor with Normalization for Random Signals of Unknown Structure

5. FUNDING NUMBERS
PE 0601152N

6. AUTHOR(S)
Albert H. Nuttall

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Undersea Warfare Center Division
1176 Howell Street
Newport, Rhode Island 02841-1708

8. PERFORMING ORGANIZATION REPORT NUMBER
NUWC-NPT TR 10,760

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Office of Naval Research
800 North Quincy Street, BCT 1
Arlington, VA 22217-5000

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

A signal (if present) is located somewhere in a band of frequencies characterized by a total of N search bins, along with uniform noise of unknown level per bin, $\underline{N}$. The signal occupies an arbitrary set of $\underline{M}$ of these bins, where not only is the extent $\underline{M}$ unknown, but, in addition, the locations of the particular $\underline{M}$ bins occupied by the signal (if present) are unknown. Also, the average signal level in an occupied bin, $\underline{S}$, is arbitrary and unknown.

In order to realize a specified false alarm probability, the power-law processor has been normalized by division with an estimate of the noise level, either from a noise-only reference or from the measured data itself. Various combinations of normalizer forms have been investigated quantitatively through their receiver operating characteristics.

14. SUBJECT TERMS
Detection
Power-Law Processor
Normalizer
Random Signals
Noise-Only Reference
Self-Reference

15. NUMBER OF PAGES
100

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	SAR

It has been found that if the number of bins, $\underline{M}$, occupied by the signal is small relative to the search size N, the additional signal-to-noise ratio required by the normalizer, in order to maintain the standard operating point, is not significant. However, if $\underline{M}$ is of the order of N/4 or larger, the degradations begin to become substantial. A partial remedy for the inherent losses caused by an unknown noise level is the use of a noise-only data reference, if available. However, eventually, as $\underline{M}$ increases and tends to N, the detection situation becomes progressively more difficult, finally becoming impossible. This is not a limit of the normalized power-law processor, but, rather, of the fact that detection of a white signal in white noise of unknown level is a theoretical impossibility.

A major problem arises with some signal processor forms when the background noise level is unknown. Namely, the actual false alarm probability realized in operation is unknown. Although the receiver operating characteristics of a particular near-optimum processor (such as the power-law processor) may indicate that good detectability performance is achievable, the actual operating point will be unknown. Changing the decision threshold may slide the operating point along a good receiver operating characteristic, but the precise location being utilized will be unknown. The normalizer forms suggested here remedy this limitation for the power-law processor by guaranteeing a prespecified false alarm probability, although at the (unavoidable) expense of a slight loss in detectability.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ACRONYMS AND SYMBOLS

CFAP	Constant False Alarm Probability
PLP	Power-Law Processor
ROC	Receiver Operating Characteristic
SNR	Signal-to-Noise Ratio
SOP	Standard Operating Point
N	Total number of search bins
$\underline{M}$	Actual number of bins occupied by signal (when present)
ν	Power law
L	Size of noise-only reference data (if available)
$\underline{N}$	Average noise level per bin
H_0	Hypothesis H_0, signal absent
H_1	Hypothesis H_1, signal present
$\underline{L}$	Actual set of bins occupied by signal
$\underline{S}$	Average signal level in an occupied bin
bold	Random variable

LIST OF ACRONYMS AND SYMBOLS (Cont'd)

x_n	Output or observation of n-th bin
q_0	Probability density of x_n under hypothesis H_0, equation (1)
q_1	Probability density of x_n under hypothesis H_1, equation (2)
P_f	False alarm probability
P_d	Detection probability
v	Fixed threshold, equations (6), (11), (13)
$X(\nu,N)$	General sample average for data $\{x_n\}$, equation (7)
$\{z_\ell\}$	Set of noise-only measurements, equation (11)
L	Size of set $\{z_\ell\}$, equation (11)
$Z(\mu,L)$	Sample average for set $\{z_\ell\}$, equation (11)
μ	Power-law used in $Z(\mu,L)$, equation (11)
P	Sum of data $\{x_n\}$ to the ν-th power, equation (15)

PERFORMANCE OF POWER-LAW PROCESSOR WITH NORMALIZATION FOR RANDOM SIGNALS OF UNKNOWN STRUCTURE

INTRODUCTION

In a series of technical reports (references 1-5), the performance of the power-law processor (PLP) for detection of random signals of unknown locations, structure, extent, and strengths has been quantified in terms of its receiver operating characteristics (ROCs) for a wide range of parameter values, such as the size of the search region, N; the number of bins occupied by the signal, $\underline{M}$; the power-law ν; and the signal-to-noise ratio (SNR) per bin. However, all of these investigations presumed that the average background noise level was flat and known, and therefore, it was normalized to unit level, without loss of generality.

For completeness and reference, the optimum processor in colored noise is derived here and found to depend on signal and noise parameters that are not likely to be known in practical applications (see appendix A). However, even if the noise powers in each and every search bin were known, they would have to be utilized in an impractical massive search over all contingencies. The impossibility of knowing all this noise level information, in addition to the astronomical number of computations that must be made, forces consideration of simpler alternatives, of which the PLP again emerges as the most reasonable practical processor. The particular power law ν is again left as a control parameter that the user can manipulate in order to maximize detection performance.

A major problem with these processor forms is that since the noise levels per bin are unknown in practice, the actual false alarm probability realized in operation is unknown. Although the ROCs of a particular PLP may indicate that good detectability performance is achievable, the actual operating point will be unknown. Changing the decision threshold may slide the operating point along a good ROC, but the precise location being utilized will be unknown.

In practice, when the average noise background is flat (or assumed flat) but of unknown level, it must be estimated and then used to select a threshold corresponding to the false alarm probability of interest. This fluctuating estimate of the noise level naturally degrades the performance of the resultant processor. Here, the losses associated with this estimation procedure, called normalization, will be quantitatively investigated in terms of the ROCs for a variety of different procedures and pertinent parameter values. The average signal powers in the occupied bins are assumed to be equal, although their level and occupancy pattern are unknown. The case of unequal signal powers per bin and known noise level did not display a strong dependence on the particular signal power set (reference 5).

Two noise-level estimation procedures will be considered, namely, the noise-only reference and the self-reference. In the former case, it is presumed that a collection of L measurements, known to consist of noise only, with the same (but unknown) average noise level as the potential signal samples, is available from which to estimate the noise level. This collection can be larger or smaller than the size N of the separate search space in which the signal is expected to reside when present. The fact that the estimated noise level fluctuates about the true value causes a degradation in performance relative to the known noise level case. This loss is unavoidable in an environment of unknown noise level.

On the other hand, in the self-reference case, no such noise-only set is available, and only the measured data (which may or may not contain a signal) must be used to establish a noise level reference. These circumstances tend to be self-defeating, because the signal (when present) biases the noise level estimate and because the finite average inherently fluctuates, both of which effects obscure some weak but valid signal contributors. Nevertheless, the need to operate in such an environment occurs frequently enough in practice that this self-reference procedure must be investigated and the losses assessed quantitatively.

The investigation of an unknown flat noise level begins with the derivation of an approximate Bayesian processor. The unknown noise level is assigned a prior density, as propounded in reference 6, chapter 2; in particular, this procedure is recommended for cases where there is insufficient knowledge of the pertinent parameters of the signal processing problem of interest. However, numerous assumptions and approximations are required to conduct the manipulations, as shown in appendix B. Adopting this procedure leaves open the possibility for numerous modifications of the basic processing technique, allowing both for simplicity in the data processing, as well as for possible performance gains.

The performances of the noise-only reference and the self-reference procedures, based upon the suggested form of the Bayesian processor, are evaluated by means of their ROCs. These procedures lead naturally to a more general form of normalizer for the self-reference cases, which is thoroughly investigated for a wide range of parameter values. The corresponding ROCs are collected in the appendixes of this report for future reference.

PROBLEM DEFINITION

The search space consists of N (frequency) bins, each containing independent, identically distributed noises of unknown average level $\underline{N}$ under hypothesis H_0, signal absent. The number of bins N is under the user's control and is always a known quantity. When signal is absent, the probability density function of each of the N bin output noises is known, except for absolute level $\underline{N}$.

When signal is present, hypothesis H_1, the quantity $\underline{M}$ is the actual number of bins occupied by the signal; most often, this is an unknown parameter. The quantity $\underline{L}$ is the actual set of bins occupied by signal components when a signal is present; for example, if $\underline{M}$ = 4, then $\underline{L}$ might be {2,3,7,29} for the occupied set, meaning that bins 2,3,7,and 29 have signal in them. This quantity $\underline{L}$ is always unknown in these investigations. Finally, the quantity $\underline{S}$ is the actual average signal level in each of the occupied bins in set $\underline{L}$ when a signal is present; this average signal level $\underline{S}$ is unknown in practice.

Nothing is presumed to be known about the received signal structure, such as whether or not it is deterministic; rather, the signal is taken to be random with no known structure. Thus, for example, the signal is not presumed to be a collection of harmonics of unknown fundamental frequency, nor must the signal occupy a contiguous band of frequencies of unknown bandwidth and/or center frequency. Instead, the signal is allowed to occupy $\underline{M}$ bins of the search band of N bins in an unspecified (nonoverlapping) independent random manner.

PROBABILITY DENSITY FUNCTIONS OF INDIVIDUAL BIN OUTPUTS

The detailed character of the two probability density functions of the available bin data, namely, q_0 and q_1, is now specified under hypotheses H_0 and H_1, respectively. In both hypotheses, the bin outputs or observations $\{x_n\}$, $1 \le n \le N$, are taken as the squared envelopes of the outputs of (disjoint) narrowband filters subject to a Gaussian random process excitation; alternatively, the observations can be interpreted as the magnitude-squared outputs of a fast Fourier transform subject to a Gaussian process input. It is assumed that these magnitude-squared bin outputs, that is, random variables $\{x_n\}$, are statistically independent of each other, which is consistent with the frequency-disjoint requirement and a Gaussian process excitation.

Since the bin output average noise level is $\underline{N}$, the probability density function of the n-th observation x_n is, under hypothesis H_0, an exponential of the form

$$q_0(u_n) = \frac{1}{\underline{N}} \exp\left(-\frac{u_n}{\underline{N}}\right) \quad \text{for } u_n > 0 \ , \quad 1 \le n \le N \ . \tag{1}$$

On the other hand, when signal is present, hypothesis H_1, the density of output x_n for this bin occupied by the m-th signal with bin output average signal level $\underline{S}$ is changed to

$$q_1(u_n) = \frac{1}{\underline{N} + \underline{S}} \exp\left(\frac{-u_n}{\underline{N} + \underline{S}}\right) \quad \text{for } u_n > 0 \ , \quad n \in \underline{L} \ . \tag{2}$$

The signal-to-noise ratio per bin is $\underline{S}/\underline{N}$.

ALTERNATIVE PROCESSOR FORMS

For known levels $\underline{N}$ and $\underline{S}$, the optimum processor is derived in appendix A, with the end result for the likelihood ratio given by equations (A-7)-(A-9). A series of necessary practical approximations reduces this general result to the standard power-law processor (A-14), namely, threshold comparison

$$\frac{1}{N}\sum_{n=1}^{N} x_n^{\nu} \gtrless T_1 \ , \tag{3}$$

where power-law ν is a control parameter to be optimized by the user and T_1 is a fixed threshold. Previous analyses of this processor (references 1-5) have revealed that its ROCs lie within a small fraction of a decibel from the optimum processor (A-7) if signal extent $\underline{M}$ is known. On the other hand, when $\underline{M}$ is unknown, power-law choice $\nu = 2.4$ performs within 1.2 dB of the optimum, regardless of the actual unknown value of $\underline{M}$. These conclusions are based upon noise and signal models (1) and (2).

When search size N and power-law ν are specified in equation (3), the false alarm probability P_f can be calculated for a given threshold T_1, provided that average noise level $\underline{N}$ is known. Conversely, for a specified P_f, the required threshold T_1 can be determined, but, again, only if average noise level $\underline{N}$ is known.

An equivalent form for processor (3) is

$$\left(\frac{1}{N}\sum_{n=1}^{N} x_n^{\nu}\right)^{1/\nu} \gtrless T_1^{1/\nu} \equiv T_2 \equiv v_2\,\underline{N} \ , \tag{4}$$

where T_2 is a fixed threshold directly proportional to the average noise level $\underline{N}$. That is, scalar v_2 is dimensionless, because the left-hand side of equation (4) is linearly proportional to $\underline{N}$. The processor in equation (4) has the same ROCs as in equation (3), because a monotonic nonlinear transformation of a decision variable does not affect the ROCs.

Finally, an alternative form to equation (4) is

$$\frac{\left(\frac{1}{N}\sum_{n=1}^{N} x_n^{\nu}\right)^{1/\nu}}{\underline{N}} \gtrless v_2 \;, \tag{5}$$

where the right-hand side is a fixed threshold, independent of $\underline{N}$. This form suggests that when average noise level $\underline{N}$ is unknown, an estimate of it should be used in the denominator of the left-hand side, while the right-hand side is kept fixed.

The problem of deriving the optimum processor for unknown signal and noise levels is undertaken in appendix B, where the use of prior densities in a Bayesian approach is employed. Several assumptions and approximations must be adopted to make significant analytic progress in the derivation of the likelihood ratio, culminating in equation (B-15). Finally, an approximation to the likelihood ratio yields the final form (B-18), namely,

$$\frac{\left(\sum_{n=1}^{N} x_n^{\nu}\right)^{1/\nu}}{\sum_{n=1}^{N} x_n} \gtrless v \;, \tag{6}$$

where v is a fixed threshold. Remarkably, this form is similar to that suggested by form (5), where noise level $\underline{N}$ was known.

NORMALIZER FORMS

DIFFERENT SAMPLE AVERAGES

Before the normalizer forms that will be considered here are specified, it is worthwhile to examine the potential of a general sample average. Consider the random variable

$$\mathbf{X}(\nu,N) \equiv \left(\frac{1}{N}\sum_{n=1}^{N} \mathbf{x}_n^{\nu}\right)^{1/\nu} . \tag{7}$$

For $\nu = 1$, $\mathbf{X}(1,N)$ is the sample mean of the observed data $\{\mathbf{x}_n\}$. For $\nu = 2$, $\mathbf{X}(2,N)$ is the sample root-mean-square value of $\{\mathbf{x}_n\}$. For $\nu = \infty$, $\mathbf{X}(\infty,N)$ is the maximum of the data set, $\max\{\mathbf{x}_1,\ldots,\mathbf{x}_N\}$. For $\nu = -1$, $\mathbf{X}(-1,N)$ is the sample harmonic mean:

$$\mathbf{X}(-1,N) = N\bigg/\sum_{n=1}^{N} \frac{1}{\mathbf{x}_n} . \tag{8}$$

Finally, as $\nu \to 0+$, the sample geometric mean is obtained as follows:

$$\mathbf{X}(\nu,N) = \left(\frac{1}{N}\sum_{n=1}^{N} \exp(\nu \ln\mathbf{x}_n)\right)^{1/\nu} \cong \left(\frac{1}{N}\sum_{n=1}^{N}(1 + \nu \ln\mathbf{x}_n)\right)^{1/\nu} =$$

$$= \left(1 + \frac{\nu}{N}\sum_{n=1}^{N} \ln\mathbf{x}_n\right)^{1/\nu} \cong \exp\left(\frac{1}{N}\sum_{n=1}^{N} \ln\mathbf{x}_n\right) = \exp\left(\frac{1}{N} \ln\prod_{n=1}^{N}\mathbf{x}_n\right) =$$

$$= \left(\prod_{n=1}^{N}\mathbf{x}_n\right)^{1/N} \quad \text{as } \nu \to 0+ . \tag{9}$$

With regard to the number of data samples, N, the limiting value of equation (7) is

$$\mathbf{X}(\nu,\infty) = \left(\overline{x_n^\nu}\right)^{1/\nu} = f(\nu)\ \underline{N} \quad \text{under } H_0 \ , \tag{10}$$

where function $f(\nu)$ is independent of $\underline{N}$. Thus, $\mathbf{X}(\nu,N)$ is a stable scaled estimate of the average noise level $\underline{N}$, when sample size N is large.

It is seen that general sample average $\mathbf{X}(\nu,N)$ defined in equation (7) encompasses a wide variety of forms. However, regardless of the value of ν, $\mathbf{X}(\nu,N)$ has the same dimensions as data $\{\mathbf{x}_n\}$. Also, a common scaling of the data according to $\{a\ \mathbf{x}_n\}$ yields scaled sample average $a\ \mathbf{X}(\nu,N)$. The data processing in equation (5) can now be expressed simply as $\mathbf{X}(\nu,N)/\underline{N}$, whereas that in equation (6) is proportional to $\mathbf{X}(\nu,N)/\mathbf{X}(1,N)$.

NOISE-ONLY REFERENCE FOR NORMALIZATION

In this case, it is presumed that a set of L noise-only, envelope-squared bin outputs $\{z_\ell\}$, $1 \le \ell \le L$, with the same (but unknown) average level $\underline{N}$ as data $\{x_n\}$ under H_0 are available. That is, the mean of each z_ℓ is equal to $\underline{N}$. This suggests an alternative to processors (5) and (6) in the form

$$\frac{X(\nu,N)}{Z(\mu,L)} \gtrless v \, , \tag{11}$$

where v is a fixed threshold and $Z(\mu,L)$ is a sample average for the L noise-only samples $\{z_\ell\}$ (see equation (7)). The parameter μ can be chosen to be equal to 1 or smaller, so as not to accentuate large z_ℓ values. Whereas large ν in the numerator of equation (11) accentuates (enhances) signal detection, a large value of μ in the denominator would tend to emphasize outliers in noise-only set $\{z_\ell\}$, which is an undesirable effect. Small μ, on the other hand, tends to suppress noise outliers in the set $\{z_\ell\}$.

The value of ratio (11) does not depend on the absolute scale of the average noise level $\underline{N}$ of data sets $\{x_n\}$ and $\{z_\ell\}$. A common scaling will be canceled in the ratio (11). Also, the case of $L = \infty$ corresponds to the previous case of known average noise level, since $Z(\mu,\infty) = f(\mu)\ \underline{N}$, where function $f(\mu)$ is independent of $\underline{N}$. Thus, a large noise-only data set is desirable, if it can be achieved.

The normalizer in equation (11) furnishes a baseline against which other normalizers can be compared for detectability performance. That is, the availability of a noise-only reference is an intermediate situation between knowledge of the average noise level $\underline{N}$ and no knowledge of $\underline{N}$. In the latter case, $\underline{N}$ must be estimated directly from the available data $\{\mathbf{x}_n\}$, without the aid of any auxiliary set of noise-only data, such as $\{z_\lambda\}$. In an intermediate case where L < N, some combination of averages may be in order for estimation of $\underline{N}$, such as

$$\alpha\, \mathbf{X}(1,N) + (1 - \alpha)\, \mathbf{Z}(1,L) \; . \tag{12}$$

SELF-REFERENCE FOR NORMALIZATION

When the average noise level $\underline{N}$ is unknown, and only the data set $\{x_n\}$ is available (which may or may not contain a signal), approximate likelihood ratio processor (6) can be generalized to the form

$$\frac{X(\nu,N)}{X(\mu,N)} \gtrless v \, , \qquad \nu > \mu \, . \tag{13}$$

The ratio in equation (13) is dimensionless and is independent of the actual absolute noise level $\underline{N}$; that is, any scaling of data set $\{x_n\}$ disappears in ratio (13). Thus, test (13) has constant false alarm probability (CFAP) behavior; that is, for given values of μ, ν, and N, it is possible to determine threshold v in equation (13) so as to exactly realize a specified P_f without knowledge of average noise level $\underline{N}$. Of course, the fact that the denominator of equation (13) is itself noisy causes a degradation in performance relative to the alternative case of known level $\underline{N}$.

If all the data values $\{x_n\}$ are equal, the ratio in equation (13) is equal to 1. On the other hand, if only one data value is nonzero, then the ratio is equal to N^b, where $b = 1/\mu - 1/\nu$. These ratios are the lowest and highest values, respectively, that ratio (13) can take on, regardless of the values of the data $\{x_n\}$.

When signal is present in $\underline{M}$ bins (of unknown location) in the set of N search bins, the detection probability P_d of normalizer (13) will depend additionally on signal-to-noise ratio $\underline{S}/\underline{N}$, as well as on $\underline{M}$. However, a cautionary note should be made. For known noise level, the optimum processor for $\underline{M} = N$ (that is, fully occupied bins) utilized $\nu = 1$. But, for $\underline{M} = N$ with flat unknown signal level and flat unknown noise level over the entire search region, there is no distinction possible between signal present versus signal absent. That is, a measurement $\{\mathbf{x}_n\}$ could have as likely occurred due to noise only as for signal plus noise. There is no information in an overall increase of the average power in data set $\{\mathbf{x}_n\}$ when all N bins are occupied by signal.

What this means is that the performance of any normalizer must deteriorate as the number of bins $\underline{M}$ occupied by signal tends to N. This is an inherent property that might be slowed by good normalizer choice, but it cannot be stopped. In the limit, if $\underline{M} = N$, reliable signal detection is impossible.

At the other extreme, where $\underline{M} << N$, normalizer (13) with $\mu = 1$ can be expected to have good performance. One reason is that equation (13) with $\mu = 1$ is exactly the form that the approximate likelihood ratio (6) takes on (see equation (B-18)). Another reason is that denominator $\mathbf{X}(1,N)$ in equation (13) will be more heavily influenced by the larger number of bin outputs that have only noise in them, $N - \underline{M}$, than by the smaller number $\underline{M}$ of bin outputs that (may) also have signal in them. Thus, a reliable estimate of average noise level $\underline{N}$ is afforded by $\mathbf{X}(1,N)$ when $\underline{M} << N$.

PERFORMANCE RESULTS

SMALL M VALUES

This subsection will consider small values for the number of bins occupied by signal, namely, $\underline{M}$ = 1, 4, 16, and 64. The total search size is N = 1024. The particular normalizer of interest here is given by equation(13) with μ = 1, namely,

$$\frac{X(\nu,N)}{X(1,N)} \gtrless v \ . \tag{14}$$

There are extreme difficulties encountered in the analysis of ratio (14). Two special cases (namely, $\nu = \infty$ and $\nu = 2$) are analyzed in appendix C for the false alarm probability. The $\nu = \infty$ case is accomplished for general N, while the $\nu = 2$ case is limited to just N = 2. Therefore, it is generally necessary to resort to simulation to determine the corresponding ROCs of the normalizers. These ROCs for equation (14) are collected in appendix D.

For the case of $\underline{M}$ = 1, only one signal bin is occupied. ROCs for $\nu = \infty$, 3, 2.5, 2, and 1+ are given in appendix D (figures D-1 through D-5, respectively); the ν = 1+ case will be explained below. These simulations of equation (14) were conducted for average noise level $\underline{N}$ = 1, without loss of generality; therefore, average signal level parameter $\underline{S}$ in appendix D can be interpreted as the SNR (in decibels) per bin.

From these ROCs, it is possible to extract the required SNR per bin to realize the standard operating point (SOP) P_f = 1E-3, P_d = 0.5, where P_d is the detection probability when signal is present. These SNR (dB) values are listed in table 1 below, not only for $\underline{M}$ = 1, but for $\underline{M}$ = 4, 16, and 64 as well. There are two numbers listed for each entry – for example, 12.8 over 12.77 for $\underline{M}$ = 1 and $\nu = \infty$. The upper number, 12.8, is the required decibel level read directly from figure D-1; thus, it is the required SNR (dB) for normalizer (14) operating in an <u>unknown</u> noise level. The lower number, 12.77, is the required SNR (dB) for the PLP operating with <u>known</u> noise level, and is obtained from reference 3, page 28 and appendixes A through C, or from reference 5, pages 53-80. The last column, OPT-B, lists the absolute lower bound on the required SNR obtained from a banding procedure in reference 4, page 33, for <u>known</u> noise level.

Table 1. Required SNR (dB) for Normalizer (14)

$\underline{M}$ \ ν	∞	3	2.5	2	1+	OPT-B
1	12.8[a] 12.77[b]	13.0 13.2*	13.2 13.8*	13.8 14.8*	16.0 21.53*	12.75[c]
4	8.2 8.14	8.05 8.05	8.2 8.4*	8.6 9.3*	10.2 14.34*	7.88
16	5.55 5.28	4.75 4.4	4.75 4.6	4.95 5.1*	6.05 8.09*	4.35
64	4.0 3.12	2.2 1.0	2.0 0.75	2.0 0.8	2.6 2.03	0.71

a: appendix D   b: reference 3, page 28   c: reference 4, page 33

Some entries are marked with a *; in these cases, the normalizer (with unknown noise level) actually requires <u>less</u> signal power than the corresponding (same ν) PLP with known noise level. However, these signal levels are still larger than the absolute minimum values for the OPT-B processor, listed in the rightmost column. Also, the <u>best</u> PLP (with known noise level) at each $\underline{M}$ always requires less SNR than the normalizer (with unknown noise level).

For $\underline{M} = 1$, the best normalizer is again $\nu = \infty$, that is, the maximum of the data set $\{x_n\}$, even though the noise level is unknown. In fact, the SNR required for unknown noise level is virtually identical to that required for known noise level, namely, 12.8 dB versus 12.77 dB. For smaller values of ν, the required SNR increases above the minimum 12.8-dB level required for $\nu = \infty$; however, it increases at a <u>slower</u> rate for the normalizer than for the PLP. For example, at $\nu = 2$, the normalizer can operate at the SOP with SNR = 13.8 dB, whereas the PLP requires 14.8 dB, even though the PLP is given more information, namely, knowledge of noise level $\underline{N}$. This behavior is a reflection of the fact that PLP $\nu = 2$ constitutes a significant mismatch with the best PLP, $\nu = \infty$, for this case of $\underline{M} = 1$, when the noise level is known.

For $\underline{M} = 4$ (figures D-6 through D-10), the best power-law value is $\nu = 3$, whether the noise level is known or not; in addition, the required SNR at the SOP is the same (that is, 8.05 dB) for both cases. Furthermore, the use of PLP $\nu = 2$, with known noise level, performs somewhat poorer than the corresponding normalizer with $\nu = 2$, namely, 9.3 dB versus 8.6 dB. This behavior continues as $\nu \to 1+$.

For $\underline{M} = 16$ (figures D-11 through D-15), the best power-law value is $\nu = 3$ for known noise level, whereas it is $\nu \cong 2.5$ for unknown noise level. Nevertheless, the known-level case requires less signal power, namely, 4.4 dB versus 4.75 dB. (This 4.4-dB level is virtually at the optimum possible level of 4.35 dB.) The loss associated with lack of knowledge of the noise level is 4.75 - 4.4 = 0.35 dB.

For $\underline{M} = 64$ (figures D-16 through D-20), the best power-law value for ν is in the range 2 to 2.5. However, the loss associated with unknown noise level has now increased to 2.0 - 0.75 = 1.25 dB. This is a manifestation of the fact that as $\underline{M}$ increases toward N, the ability to determine the presence of a flat-topped signal spectrum in white noise tends to zero, with the case of $\underline{M} = N$ being absolutely impossible when the noise level is unknown.

When $\nu = 1$ in equation (14), the left-hand side is equal to 1, independent of the data $\{\mathbf{x}_n\}$. However, if ν is allowed to approach 1 from above, a meaningful CFAP processor evolves. More generally, in appendix E, the limit of normalizer (13), as $\mu \to \nu$ from below, is derived. The end result is equation (E-5), namely,

$$\frac{1}{N} \sum_{n=1}^{N} \frac{\mathbf{x}_n^\nu}{\mathbf{P}/N} \ln\left(\frac{\mathbf{x}_n^\nu}{\mathbf{P}/N}\right) \gtrless \mathrm{v} \; , \quad \text{where} \quad \mathbf{P} = \sum_{n=1}^{N} \mathbf{x}_n^\nu \; . \tag{15}$$

This processor, with ν now set equal to 1, is the one simulated and tabulated in table 1 under the heading $\nu = 1+$. Processor (15) has CFAP capability regardless of the value of ν, because replacement of $\{x_n\}$ by $\{a\ x_n\}$ yields the same output, independent of scale factor a.

In summary of this subsection, the losses associated with the self-reference normalizer of equation (14) with N = 1024 at the SOP are not significant for the number of signal bins $\underline{M} \leq 64$. In fact, the performance of some of the normalizers in table 1 is a little better than the PLP and is less sensitive to the actual value of the power-law ν that is used.

NUMBER OF SIGNAL BINS $\underline{M}$ = 256

This subsection deals solely with the case of $\underline{M}$ = 256 and N = 1024. (There is no need to consider $\underline{M}$ = N = 1024, since that is an impossible detection scenario, namely, flat signal in flat noise of unknown level.) The normalizer of interest is now the more general self-reference normalizer in equation (13), namely,

$$\frac{\mathbf{X}(\nu,N)}{\mathbf{X}(\mu,N)} \gtrless v \ , \qquad \nu > \mu \ . \tag{16}$$

Again, in order to determine the ROCs of processor (16), it is necessary to resort to simulation; these ROCs are collected together in appendix F. A thorough search of pairs of values for parameters ν and μ was conducted in the neighborhood of the point where the best performance was obtained. The required SNR (dB) to realize the SOP $P_f = 0.001$, $P_d = 0.5$ at each ν,μ pair was extracted from the ROCs and is given in table 2 below. The two regions $\nu < \mu$ and $\mu < 0$ were also briefly considered, but yielded ROCs that had P_d less than P_f; since this situation is totally unacceptable, these cases are not presented. The cases in table 2 where $\mu = \nu$ are actually for the limiting normalizer of appendix E (namely, equation (E-5)), where $\mu \to \nu$ from below.

Observation of table 1 reveals that the pair $\nu = 1.5$, $\mu = 1$ yields performance as good as any other combination. However, switching to the more convenient pair $\nu = 2$, $\mu = 1$ loses only a small fraction of a decibel and would probably be adopted in practice. Also, reference to the PLP results for known noise

Table 2. Required SNR (dB) for Normalizer (16)

$\mu \backslash \nu$	3.0	2.5	2.0	1.75	1.5	1.25	1.0
0	0.3	0.2	0.3		0.7		1.5
0.5	0.2	0.0	-.08		0.12		0.5
1.0	0.4	0.15	-.08	-0.1	-0.15	-0.1	0.07
1.25				-0.05	-0.15	-0.15	
1.5	1.0	0.5	0.15	0.0	-0.1		
1.75				0.15			
2.0			0.5				

level reveals that the two power-law choices, $\nu = 2$ and $\nu = 1$, both have equal performance at $\underline{M} = 256$ when $N = 1024$ (see, for example, reference 4, figure 22).

The required values of SNR (dB) at the SOP for the PLP with known noise level, for $\underline{M} = 256$ and $N = 1024$, are given in reference 3, page 28, table 1. For example, both $\nu = 1$ and $\nu = 2$ require -4.0 dB, while $\nu = 2.5$ requires -3.6 dB. Both of these decibel levels are substantially better than the best of the results in table 2 above, namely, -0.15 dB. Thus, the self-reference normalizer of equation (16) suffers at least a 3.85-dB degradation when such a large fraction of the search space is occupied by signal, that is, 256 bins out of 1024. This does not mean that equation (16) specifies a poor processor; rather, it means that the additional lack of knowledge of the average noise level may result in permanent and irretrievable losses.

NOISE-ONLY REFERENCE FOR NORMALIZATION

In an attempt to determine whether the losses above are irretrievable or not, the performance of a noise-only reference for normalization was investigated, namely, processor (11) for $\underline{M} = 256$ and $L = N = 1024$:

$$\frac{\mathbf{X}(\nu,N)}{\mathbf{Z}(\mu,L)} \gtrless v \; . \tag{17}$$

size L could be larger or smaller than N; L = N is used here, both for convenience and for a direct comparison of results with the alternative denominator $\mathbf{X}(\mu,N)$ used earlier in equation (16).

The ROCs for equation (17) with $\mu = 1$ are presented in appendix G. The required values for SNR (dB) at the SOP are listed in table 3 below for five values of ν. The smallest required signal level of -2.7 dB is achieved for $\nu = 2$; this level is 1.3 dB poorer than the -4.0-dB SNR required for the corresponding $\nu = 2$ PLP operating in a known noise level (see reference 4, page 33). This loss of 1.3 dB for normalizer (17) is very reasonable when it is considered that, despite the lack of knowledge of the average noise level, the specified false alarm probability of 0.001 is guaranteed by processor (17) with $\mu = 1$.

An additional case for normalizer (17) was numerically investigated, namely, $\mu \to 0+$. Recall from equation (9) that the denominator of equation (17) is then the geometric mean of the available noise-only data $\{z_\ell\}$. The corresponding ROCs are

Table 3. Required SNR (dB) for Normalizer (17) with $\mu = 1$

ν	1	1.5	2	2.5	3
SNR (dB)	-2.3	-2.6	-2.7	-2.65	-2.4

presented in appendix G, and the required values for SNR (dB) at the SOP are listed in table 4 below.

The best case for a geometric-mean reference requires SNR (dB) = −2.2 dB, which is 0.5 dB poorer than the best case for the arithmetic mean in table 3. These results were obtained for $\underline{M}$ = 256 and L = N = 1024.

Table 4. Required SNR (dB) for Normalizer (17) with μ = 0+

ν	1	1.5	2	2.5	3
SNR (dB)	−1.6	−1.9	−2.15	−2.2	−2.15

SUMMARY

The possibility of modifying the power-law processor so that it can maintain its high-quality detection capability in a noise environment of unknown level, and yet realize a specified false alarm probability, has been investigated for a number of normalizers. It has been found that if the number of bins, $\underline{M}$, occupied by signal is small relative to the search size N, the additional signal-to-noise ratio required to maintain the standard operating point is not significant. However, if $\underline{M}$ is of the order of N/4 or larger, the degradations begin to become substantial. A partial remedy to the inherent losses caused by unknown noise level is to use a noise-only data reference, if available. However, eventually, as $\underline{M}$ increases and tends to N, the detection situation becomes progressively more difficult, finally becoming impossible. This is not a limit of the power-law processor, but, rather, of the fact that detection of a white signal in white noise of unknown level is a theoretical impossibility.

Some of the normalizers come very close to the power-law processor performance, despite an unknown noise level. For example, for N = 1024, $\underline{M}$ = 16, and ν = 3, the normalizer requires a signal-to-noise ratio of 4.75 dB, whereas the power-law processor requires 4.4 dB, a difference of only 0.35 dB. Furthermore, the absolute optimum requirement by the banding processor that knows and utilizes $\underline{M}$, $\underline{S}$, and $\underline{N}$ is 4.35 dB. It is quite remarkable that the normalizer is only 0.4 dB poorer than

this optimum level, considering that the normalizer is completely ignorant of $\underline{M}$, $\underline{S}$, and $\underline{N}$.

There are cases where the normalizer outperforms the power-law processor when the same value of ν is used for both. However, if the best normalizer is compared with the best power-law processor, this never happens; here, best is meant in the sense of the optimum values of ν for maximum detection probability of each processor.

It appears that the performance level achieved by the normalizer is less sensitive to the exact value of ν used for signal detection than is that of the power-law processor. For example, at $\underline{M} = 64$, the three normalizers with $\nu = 2$, 2.5, and 3 all require a signal-to-noise ratio of approximately 2 dB. Also, this requirement is only 1.3 dB poorer than the optimum level of 0.71 dB.

Some additional alternatives to the general sample average in equation (7) were considered for use in the denominator of normalizer (13) instead of $\mathbf{X}(\mu,N)$. The first was the median of the measured data $\{\mathbf{x}_n\}$; the second was the sample arithmetic mean of a central section of the ordered random variables $\mathbf{x}'_1 \geq \mathbf{x}'_2 \geq \cdots \geq \mathbf{x}'_N$; and the third was the sample geometric mean of a central section of the ordered random variables. Despite the size and location of the sections, none of these modified normalizers utilizing ordered data outperformed the normalizers considered here. Thus, it appears that the normalizer with $\nu = 2$ and $\mu = 1$ performs about as well as is possible.

APPENDIX A — OPTIMUM PROCESSOR FOR COLORED NOISE

Let $\underline{N}_n$ be the mean noise level in bin n. Then, the joint probability density function governing the observation $\{x_n\}$, $1 \le n \le N$, under the noise-only hypothesis H_0 is

$$p_0(u_1,\ldots,u_N) = \prod_{n=1}^{N} \left\{ \frac{1}{\underline{N}_n} \exp\left(\frac{-u_n}{\underline{N}_n} \right) \right\} \quad \text{for all } u_n > 0 . \tag{A-1}$$

On the other hand, under hypothesis H_1, when bin n is also occupied by the m-th signal with mean signal level $\underline{S}_m$, the probability density function of this particular bin output is

$$\frac{1}{\underline{S}_m + \underline{N}_n} \exp\left(\frac{-u_n}{\underline{S}_m + \underline{N}_n} \right) = \underline{a}_{mn} \exp(- \underline{a}_{mn} u_n) \quad \text{for } u_n > 0 , \tag{A-2}$$

where the strength parameters are defined as

$$\underline{a}_{mn} = \frac{1}{\underline{S}_m + \underline{N}_n} \quad \text{for } 1 \le m \le \underline{M} , \quad 1 \le n \le N . \tag{A-3}$$

This leads to the joint probability density function governing the observation $\{x_n\}$ under hypothesis H_1 in the form

$$p_1(u_1,\ldots,u_N) = \underset{\text{no two equal}}{\sum_{j=1}^{N} \sum_{k=1}^{N} \cdots \sum_{\ell=1}^{N}} \left[\frac{1}{K_0} \underline{a}_{1j} \exp\left(- \underline{a}_{1j} u_j\right) \times \right.$$

$$\left. \times \underline{a}_{2k} \exp\left(- \underline{a}_{2k} u_k\right) \times \cdots \times \underline{a}_{\underline{M}\ell} \exp\left(- \underline{a}_{\underline{M}\ell} u_\ell\right) \prod_{\substack{n=1 \\ n \ne j,k,\ldots,\ell}}^{N} \left\{ \frac{1}{\underline{N}_n} \exp\left(\frac{-u_n}{\underline{N}_n} \right) \right\} \right] ; \tag{A-4}$$

the "no two equal" qualifier under the summations means that none of the integers $j,k,\ldots,\ell$ can be equal to each other, while the slash on the product indicates that $n \neq j,k,\ldots,\ell$ is required. There are $\underline{M}$ summations in equation (A-4). The constant K_0 is given by $K_0 = N(N-1)\cdots(N+1-\underline{M})$, since none of the $\underline{M}$ occupied signal bins can overlap. This quantity K_0 is also the total number of terms in the $\underline{M}$ summations in equation (A-4), all of which possibilities can occur with equal probability $1/K_0$.

The likelihood ratio for observation $\{x_n\}$, $1 \le n \le N$, is given, upon use of equations (A-1) and (A-4), by the random variable

$$\mathbf{LR} \equiv \frac{p_1(x_1,\ldots,x_N)}{p_0(x_1,\ldots,x_N)} = \frac{1}{K_0} \underbrace{\sum_{j=1}^{N} \sum_{k=1}^{N} \cdots \sum_{\ell=1}^{N}}_{\text{no two equal}} \frac{\underline{N}_j}{\underline{S}_1 + \underline{N}_j} \, \frac{\underline{N}_k}{\underline{S}_2 + \underline{N}_k} \times \cdots$$

$$\times \frac{\underline{N}_\ell}{\underline{S}_{\underline{M}} + \underline{N}_\ell} \exp\left(\underline{w}_{1j}\, x_j + \underline{w}_{2k}\, x_k + \cdots + \underline{w}_{\underline{M}\ell}\, x_\ell\right) , \qquad \text{(A-5)}$$

where the weights have been defined as

$$\underline{w}_{mn} \equiv \frac{1}{\underline{N}_n} \frac{\underline{S}_m}{\underline{S}_m + \underline{N}_n} \quad \text{for } 1 \le m \le \underline{M} \ , \ 1 \le n \le N \ . \qquad \text{(A-6)}$$

Compare the form of this weighting with that in reference 7, page 112, equation (420), or page 488, equation (78). The leading factor in $\underline{w}_{mn}$, $1/\underline{N}_n$, normalizes (whitens) the noise levels, while the remaining factor contributes a signal-to-noise ratio emphasis.

The required data processing in equation (A-5) indicates that the following procedure is to be used: first assume that signal

1 is in bin j and weight that output $\mathbf{x}_j$ by $\underline{w}_{1j}$; also, assume that signal 2 is in bin k and weight that output $\mathbf{x}_k$ by $\underline{w}_{2k}$; then continue through signal $\underline{M}$, assumed to be in bin λ. Next, sum these particular $\underline{M}$ weighted-data quantities and exponentiate. Finally, sum over all the K_0 disjoint possibilities for $j,k,\ldots,\lambda$, using the $\underline{M}$ additional scale factors $\{\underline{N}_n/(\underline{S}_m + \underline{N}_n)\}$, as indicated in each term of equation (A-5).

If the noise is white across the search band (that is, $\underline{N}_n = \underline{N}$ for $1 \leq n \leq N$), then equations (A-5) and (A-6) can be simplified to

$$\mathbf{LR} = \frac{1}{K_0} \frac{\underline{N}}{\underline{S}_1 + \underline{N}} \frac{\underline{N}}{\underline{S}_2 + \underline{N}} \cdots \frac{\underline{N}}{\underline{S}_{\underline{M}} + \underline{N}} \times$$

$$\times \underbrace{\sum_{j=1}^{N} \sum_{k=1}^{N} \cdots \sum_{\lambda=1}^{N}}_{\text{no two equal}} \exp\left(\underline{w}_1 \mathbf{x}_j + \underline{w}_2 \mathbf{x}_k + \cdots + \underline{w}_{\underline{M}} \mathbf{x}_\lambda\right) , \qquad \text{(A-7)}$$

where the modified weights are defined as

$$\underline{w}_m = \frac{1}{\underline{N}} \frac{\underline{S}_m}{\underline{S}_m + \underline{N}} \quad \text{for } 1 \leq m \leq \underline{M} . \qquad \text{(A-8)}$$

This result for the likelihood ratio is a slight generalization of reference 5, equations (4) and (7), to nonunity noise levels, $\underline{N}$, per bin.

In addition, if the signal levels per bin are all equal, $\underline{S}_m = \underline{S}$ for $1 \leq m \leq \underline{M}$, then the weights $\{\underline{w}_m\}$ are all equal to

$$\underline{w} = \frac{1}{\underline{N}} \frac{\underline{S}}{\underline{S} + \underline{N}} . \qquad \text{(A-9)}$$

APPROXIMATION TO GENERAL LIKELIHOOD RATIO

The exponential in the general result of equation (A-5) greatly accentuates the largest of the weighted data terms, which tends to dominate the leading scale factors. Therefore, concentration is on these data-dependent terms in this equation.

If each exponential in equation (A-5) is expanded in a power series, there will be linear terms, quadratic terms, cubic terms, etc. The linear terms will contain random variable x_1 for numerous combinations of indexes $j,k,\ldots,\ell$. As a result, x_1 will be weighted by a linear sum of $\{\underline{w}_{m1}\}$ for $1 \le m \le \underline{M}$. Similarly, data value x_2 will be weighted by a linear sum of $\{\underline{w}_{m2}\}$. Thus, a reasonable approximation to the data processing of the linear terms in the expansion of equation (A-5) is, using equation (A-6),

$$z_1 \equiv \sum_{n=1}^{N} x_n \left(\sum_{m=1}^{\underline{M}} \underline{w}_{mn} \right) = \sum_{n=1}^{N} \frac{x_n}{\underline{N}_n} \sum_{m=1}^{\underline{M}} \frac{\underline{S}_m}{\underline{S}_m + \underline{N}_n} . \qquad \text{(A-10)}$$

With regard to the quadratic terms resulting from the expansion of equation (A-5), observe that random variable x_1^2 will be weighted by a sum of terms involving $\{\underline{w}_{m1}^2\}$. Therefore, an approximation to the quadratic terms is given by

$$z_2 \equiv \sum_{n=1}^{N} x_n^2 \left(\sum_{m=1}^{\underline{M}} \underline{w}_{mn}^2 \right) = \sum_{n=1}^{N} \left(\frac{x_n}{\underline{N}_n} \right)^2 \sum_{m=1}^{\underline{M}} \left(\frac{\underline{S}_m}{\underline{S}_m + \underline{N}_n} \right)^2 . \qquad \text{(A-11)}$$

Finally, for the general ν-th power, a similar argument leads to consideration of the following power-law processor:

$$z_\nu \equiv \sum_{n=1}^{N} x_n^\nu \left(\sum_{m=1}^{\underline{M}} \underline{w}_{mn}^\nu\right) = \sum_{n=1}^{N} \left(\frac{x_n}{\underline{N}_n}\right)^\nu \sum_{m=1}^{\underline{M}} \left(\frac{\underline{S}_m}{\underline{S}_m + \underline{N}_n}\right)^\nu . \tag{A-12}$$

The choice of the best value of power-law ν depends on the exact values of the parameters in the application of interest.

Notice that if the noise is white across the search band, that is, $\underline{N}_n = \underline{N}$ for $1 \le n \le N$, the inner summation on m becomes independent of n, regardless of the value of power ν. Therefore, this quantity, along with the $\underline{N}_n$ denominator factor, can be removed from the outer summation on n, leaving the standard power-law processor, which simply adds up all the data values x_n^ν.

In fact, the unlikelihood of knowing all the parameters required for implementation of the (nonwhite) processors in equations (A-10)-(A-12) forces the adoption of just such simpler procedures in practice. Thus, one possibility, in the absence of signal level information, is to replace the sum on m in equation (A-12) by a constant (independent of n), ending up with the scaled (whitened) power-law processor

$$z_\nu = \sum_{n=1}^{N} \left(\frac{x_n}{\underline{N}_n}\right)^\nu . \tag{A-13}$$

Alternatively, since all the actual noise levels $\{\underline{N}_n\}$ are also likely to be unknown, the standard power-law processor,

$$z_\nu = \sum_{n=1}^{N} x_n^\nu , \tag{A-14}$$

which requires no knowledge at all of signal or noise levels, can be used.

A problem with all these processors is that the noise levels per bin are unknown in practice. Therefore, the actual false alarm probability realized in operation is unknown. Although the ROCs may indicate that good performance is achievable, the actual operating point will be unknown. Changing the decision threshold may slide the operating point along a good ROC, but the precise location being utilized will be unknown. The only way to determine and set the false alarm probability is to estimate the noise level and use it in a modified processor form.

APPENDIX B — DERIVATION OF APPROXIMATE BAYESIAN PROCESSOR

This appendix relies heavily upon the Bayesian methods of deriving the optimum processor, as outlined in reference 6, chapter 2, where the use of prior densities is heavily propounded for cases of insufficient knowledge of the pertinent parameters of the signal processing problem of interest. Also, for future reference, frequent use of the integral result

$$\int_0^\infty dx\ x^{-1-\alpha} \exp(-\beta/x) = \frac{\Gamma(\alpha)}{\beta^\alpha} \quad \text{for } \alpha > 0 \text{ , } \beta > 0 \qquad \text{(B-1)}$$

will be made.

AVERAGE OF THE NOISE-ONLY JOINT PROBABILITY DENSITY

Consideration here is limited to the case of equal (unknown) noise levels in each bin, that is, $\underline{N}_n = \underline{a}$ for $1 \le n \le N$ in equation (13). There follows the conditional joint probability density function

$$p_0(u_1,\ldots,u_N|\underline{a}) = \underline{a}^{-N} \exp(-T/\underline{a}) \text{ , } \quad T = u_1 + \cdots + u_N \text{ , } \qquad \text{(B-2)}$$

where the unknown noise level $\underline{a}$ is considered a random variable.

The prior probability density function for the noise level (see reference 6, section 2.4) is taken as

$$p_0(\underline{a}) = A\ \underline{a}^{\mu-1} \quad \text{for } a_1 < \underline{a} < a_2 \text{ , } \quad A = \frac{\mu}{a_2^\mu - a_1^\mu} \text{ .} \qquad \text{(B-3)}$$

Here, a_1 and a_2 are hyperparameters (reference 6, page 11). Choosing $\mu = 0$ results in Jeffrey's prior (reference 6, equations (2.14) and (A.3)); however, it is found that a_1, a_2, and μ need not be specified precisely here.

The unconditional joint density under hypothesis H_0 is

$$p_0(u_1,\ldots,u_N) = \int d\underline{a}\ p_0(\underline{a})\ p_0(u_1,\ldots,u_N|\underline{a}) =$$

$$= A \int_{a_1}^{a_2} d\underline{a}\ \underline{a}^{\mu-1-N} \exp(-T/\underline{a}) \cong A \int_0^{\infty} d\underline{a}\ \underline{a}^{\mu-1-N} \exp(-T/\underline{a}) =$$

$$= A \frac{\Gamma(N-\mu)}{T^{N-\mu}} \quad \text{for } N > \mu \ . \tag{B-4}$$

This approximation to the integral for broad hyperparameters a_1 and a_2 is used for the sake of simplicity in the end result; see also reference 6, page 21, as well as equation (2.28) with equation (2.14). The result in equation (B-4) will be used in the denominator of the likelihood ratio.

AVERAGE OF THE SIGNAL-PLUS-NOISE JOINT PROBABILITY DENSITY

As above, consideration here is limited to the case of equal (unknown) signal levels in each of the $\underline{M}$ occupied bins; that is, $\underline{S}_m = \underline{b}$ for $1 \le m \le \underline{M}$ in equation (15). Then, the general j,k,... term in equation (15) becomes, upon use of equation (15) with $\underline{N}_n = \underline{a}$,

$$p_{jk}(u_1,\ldots,u_N|\underline{a},\underline{b}) \equiv \frac{K_1}{(\underline{a}+\underline{b})^{\underline{M}}} \exp\left(-\frac{T_{jk}}{\underline{a}+\underline{b}}\right) \frac{1}{\underline{a}^{N-\underline{M}}} \exp\left(-\frac{T - T_{jk}}{\underline{a}}\right) , \tag{B-5}$$

where $K_1 = 1/K_0$ is a constant (independent of $\underline{a}$, $\underline{b}$, and $\{u_n\}$) and

$$T_{jk} = T_{jk\ldots} \equiv u_j + u_k + \cdots + u_\ell \; . \tag{B-6}$$

This latter sum contains $\underline{M}$ terms. The complete joint conditional probability density function follows from equation (15) as the sum over all $j,k,\ldots,\ell$, and is denoted by $j,k,..$ according to

$$p_1(u_1,\ldots,u_N|\underline{a},\underline{b}) = \frac{K_1}{\underline{a}^{N-\underline{M}}\ (\underline{a}+\underline{b})^{\underline{M}}} \exp(-T/\underline{a}) \sum_{jk..} \exp\left(\frac{\underline{b}\ T_{jk}}{\underline{a}(\underline{a}+\underline{b})}\right) . \tag{B-7}$$

At this point, the latter exponential in equation (B-7) must be simplified in order to make any analytic progress. In keeping with the earlier approach in reference 5, pages 13-22, the power-law approximation $\exp(x) \cong K_2\, x^\nu$ is again used, where power $\nu > 0$ is yet to be chosen. Then, the sum in equation (B-7) becomes

$$\frac{K_2\ \underline{b}^\nu}{\underline{a}^\nu\ (\underline{a}+\underline{b})^\nu} \sum_{jk..} T_{jk}^\nu \equiv \frac{K_2\ \underline{b}^\nu}{\underline{a}^\nu\ (\underline{a}+\underline{b})^\nu} \Sigma_\nu \; . \tag{B-8}$$

The power-law approximation allows the separation of the variables $\underline{a}$ and $\underline{b}$ from the arguments $\{u_n\}$, and is the crucial step in this derivation. Substitution of equation (B-8) in equation (B-7) yields the approximation for the probability density function as

$$p_1(u_1,\ldots,u_N|\underline{a},\underline{b}) = \frac{K_1 \; K_2 \; \underline{b}^{\nu} \exp(-T/\underline{a})}{\underline{a}^{N-\underline{M}+\nu} \; (\underline{a} + \underline{b})^{\underline{M}+\nu}} \; \Sigma_\nu \; . \tag{B-9}$$

A prior probability density function is now assumed for the signal level, which is of the same form as was taken for the noise level in equation (B-3), namely,

$$p_1(\underline{b}) = B \; \underline{b}^{\gamma-1} \quad \text{for } b_1 < \underline{b} < b_2 \; , \quad B = \frac{\gamma}{b_2^\gamma - b_1^\gamma} \; . \tag{B-10}$$

Again, $\gamma = 0$ corresponds to Jeffrey's prior (reference 6, equation (2.14)) for the signal level. The unconditional joint density under hypothesis H_1 is given by

$$p_1(u_1,\ldots,u_N) = \iint d\underline{a} \; d\underline{b} \; p_0(\underline{a}) \; p_1(\underline{b}) \; p_1(u_1,\ldots,u_N|\underline{a},\underline{b}) \; . \tag{B-11}$$

When equations (B-9) and (B-10) are employed in equation (B-11), the integral over $\underline{b}$ takes the form

$$\int_{b_1}^{b_2} d\underline{b} \; B \; \underline{b}^{\gamma-1} \frac{\underline{b}^{\nu}}{(\underline{a} + \underline{b})^{\underline{M}+\nu}} \cong \int_0^\infty d\underline{b} \; B \; \frac{\underline{b}^{\nu+\gamma-1}}{(\underline{a} + \underline{b})^{\underline{M}+\nu}} =$$

$$= B \; \underline{a}^{\gamma-\underline{M}} \int_0^\infty \frac{dx \; x^{\nu+\gamma-1}}{(1 + x)^{\underline{M}+\nu}} = B \; K_3 \; \underline{a}^{\gamma-\underline{M}} \quad \text{for } \nu + \gamma > 0 \; , \; \underline{M} > \gamma. \tag{B-12}$$

Here, $K_3 = \Gamma(\nu+\gamma) \; \Gamma(\underline{M}-\gamma)/\Gamma(\underline{M}+\nu)$. The remaining integral on $\underline{a}$ then yields

$$p_1(u_1,\ldots,u_N) = \int_{a_1}^{a_2} d\underline{a} \; A \; \underline{a}^{\mu-1} \; B \; K_3 \; \underline{a}^{\gamma-\underline{M}} \; \frac{K_1 \; K_2 \; \exp(-T/\underline{a})}{\underline{a}^{N-\underline{M}+\nu}} \; \Sigma_\nu$$

$$\cong A\ B\ K_1\ K_2\ K_3 \int_0^\infty d\underline{a}\ \underline{a}^{\mu+\gamma-1-\nu-N}\ \exp(-T/\underline{a})\ \Sigma_\nu =$$

$$= A\ B\ K_1\ K_2\ K_3\ \frac{\Gamma(N+\nu-\mu-\gamma)}{T^{N+\nu-\mu-\gamma}}\ \Sigma_\nu \quad \text{for } N+\nu-\mu-\gamma > 0. \tag{B-13}$$

LIKELIHOOD RATIO

The ratio of probability density functions under hypotheses H_1 and H_0 follows from equations (B-13) and (B-4) as

$$\frac{p_1(u_1,\ldots,u_N)}{p_0(u_1,\ldots,u_N)} \propto \frac{\Sigma_\nu}{T^{\nu-\gamma}} = \frac{\sum\limits_{jk..} T_{jk}^{\nu}}{T^{\nu-\gamma}} = \frac{\sum\limits_{jk..} (u_j + u_k + \cdots + u_\lambda)^\nu}{\left(\sum\limits_{n=1}^{N} u_n\right)^{\nu-\gamma}}\ , \tag{B-14}$$

where all irrelevant constants have been dropped and equations (B-8), (B-6), and (B-2) were used. With a given data set of random observations $\{\mathbf{x}_n\}$ for $1 \le n \le N$, the likelihood ratio is

$$\mathbf{LR} \equiv \frac{p_1(\mathbf{x}_1,\ldots,\mathbf{x}_N)}{p_0(\mathbf{x}_1,\ldots,\mathbf{x}_N)} \propto \frac{\sum\limits_{jk..} (\mathbf{x}_j + \mathbf{x}_k + \cdots + \mathbf{x}_\lambda)^\nu}{\left(\sum\limits_{n=1}^{N} \mathbf{x}_n\right)^{\nu-\gamma}}\ , \tag{B-15}$$

where each sum in the numerator contains $\underline{M}$ terms.

APPROXIMATION TO LIKELIHOOD RATIO

If the approximation used for the simplification of the exponential in progressing from equation (B-7) to (B-8) were taken according to power law $\nu = 1$, the numerator of equation (B-15) would consist of a scaled version of the linear sum of all the data $\{x_n\}$.

On the other hand, if ν were taken as 2, there would be a sum of all the squares of the data and a sum of all the cross-products of the form $x_j\, x_k$ for $j \neq k$. Since all possible cross-products appear with an identical scale factor (reference 3, page 10), a reasonably good approximation to their sum is a scaled version of the sum of the squares of the data. Thus, the overall approximation to the numerator of equation (B-15) for $\nu = 2$ is simply a scaled version of the sum of the squares of all the data $\{x_n\}$.

For $\nu = 3$, a similar argument leads to the approximation of the numerator of equation (B-15) by a scaled version of the sum of the cubes of all the data. This is believed to be a rather good approximation, especially for large N, which is the total number of data points. The end result, for general power-law ν, is the approximate likelihood ratio adopted here, namely,

$$\mathrm{LR} \propto \frac{\sum_{n=1}^{N} x_n^{\nu}}{\left(\sum_{n=1}^{N} x_n\right)^{\nu-\gamma}} . \qquad \text{(B-16)}$$

It is very important to observe that equation (B-16) does not require knowledge of the signal level, nor does it utilize any information about $\underline{M}$, the number of bins occupied by signal.

However, there is one significant drawback of the processor form in equation (B-16). Even when noise alone is present, the level of the right-hand side varies with the noise level to the γ power, which is unknown. Thus, it is generally impossible to choose a threshold with processor (B-16) that has a specified false alarm probability, when the actual noise level is unknown.

However, suppose that signal parameter value $\gamma = 0$ is assumed in equation (B-10). Then, the prior probability density function for the signal level $\underline{S}$ takes the particular form

$$p_1(\underline{b}) = B/\underline{b} \quad \text{for } b_1 < \underline{b} < b_2 \;, \quad B = \frac{1}{\ln(b_2/b_1)} \;, \tag{B-17}$$

which is Jeffrey's prior (reference 6, equations (2.14) and (A.3)). It corresponds to a uniform distribution over different scales; that is, decibel measure log($\underline{S}$) is uniformly distributed (reference 6, page 11). This is an example of a convenience prior (reference 6, page 12) adopted for purposes of simplifying the processor form.

When $\gamma = 0$ is assumed in equation (B-16), the approximate likelihood ratio test on data $\{x_n\}$ takes the final form

$$\frac{\sum_{n=1}^{N} x_n^{\nu}}{\left(\sum_{n=1}^{N} x_n\right)^{\nu}} \gtrless v \;, \quad \text{or} \quad \frac{\left(\sum_{n=1}^{N} x_n^{\nu}\right)^{1/\nu}}{\sum_{n=1}^{N} x_n} \gtrless v \;, \tag{B-18}$$

where v is a fixed threshold. For noise alone, the value of the left-hand side of test (B-18) is independent of the actual noise level, since a common scalar on the data $\{\mathbf{x}_n\}$ cancels out of the ratio, regardless of power-law ν employed. Thus, a prespecified false alarm probability can be realized by proper choice of v, which depends on N, ν, and the noise statistics, but not its absolute level. Notice also that test (B-18) is independent of the noise probability density parameter μ in equation (B-3), as well as the hyperparameters a_1, a_2, b_1, and b_2. Of course, power-law value ν can and should be varied to realize optimum detection performance.

When all the conditions that the parameters have to satisfy in the above derivations are collected, they can be simply stated as

$$N > \mu \ , \quad \nu > 0 \ , \quad \underline{M} > 0 \ . \tag{B-19}$$

In general, these restrictions are easily satisfied.

ALTERNATIVE DERIVATION OF p_1

The power-law approximation made below equation (B-7) can be deferred and the likelihood ratio can be evaluated in a more accurate form. The unconditional joint probability density function is given by equation (B-11), with the integral limits dictated by equations (B-10) and (B-3). The integral on $\underline{b}$ is convergent at the lower limit b_1 for any γ. However, when the approximation is made to extend the lower limit to 0, the new integral only converges for $\gamma > 0$; this latter restriction is artificially introduced by the approximation procedure and should be ignored when possible.

If the substitution $x = \underline{b}/(\underline{a} + \underline{b})$ is made in the $\underline{b}$ integral and if reference 8, equation (3.383 1), is used, the x integral can be carried out to yield $\gamma^{-1}\ {}_1F_1(\gamma;\underline{M};T_{jk}/\underline{a})$. Then, by making the substitution $t = T_{jk}/\underline{a}$ in the resultant $\underline{a}$ integral and using reference 8, equation (7.621 4), the t integral can also be evaluated to yield $T^{\mu+\gamma-N}\ \Gamma(N - \mu - \gamma)\ {}_2F_1(\gamma, N - \mu - \gamma;\underline{M};T_{jk}/T)$. The end result is

$$p_1(u_1,\ldots,u_N) = \frac{K_1\ A\ B}{\gamma}\ \frac{\Gamma(N - \mu - \gamma)}{T^{N-\mu-\gamma}} \sum_{jk..} {}_2F_1(\gamma, N - \mu - \gamma;\underline{M};T_{jk}/T), \tag{B-20}$$

and the ratio of densities is

$$\frac{p_1}{p_0} = \frac{K_1\ B\ \Gamma(N - \mu - \gamma)}{\gamma\ \Gamma(N - \mu)}\ T^{\gamma} \sum_{jk..} {}_2F_1(\gamma, N - \mu - \gamma;\underline{M};T_{jk}/T)\ . \tag{B-21}$$

The likelihood ratio is proportional to

$$\mathbf{LR} \propto \mathbf{T}^{\gamma} \sum_{jk..} {}_2F_1(\gamma, N - \mu - \gamma; \underline{M}; \mathbf{T}_{jk}/\mathbf{T}) \ , \tag{B-22}$$

where random variables

$$\mathbf{T} = \sum_{n=1}^{N} \mathbf{x}_n \ , \qquad \mathbf{T}_{jk} = \mathbf{x}_j + \mathbf{x}_k + \cdots + \mathbf{x}_\ell \ . \tag{B-23}$$

Unfortunately, practical realization of this processor in equation (B-22) is impossible due to the inordinate number of terms required. Expansion of ${}_2F_1$ yields

$$\mathbf{LR} \propto \mathbf{T}^{\gamma} \sum_{\nu=0}^{\infty} \frac{(\gamma)_\nu \ (N - \mu - \gamma)_\nu}{(\underline{M})_\nu \ \nu!} \sum_{jk..} \left(\frac{\mathbf{T}_{jk}}{\mathbf{T}}\right)^{\nu} \ . \tag{B-24}$$

The fundamental building block of this optimum processor is $\Sigma\ \mathbf{T}_{jk}^{\nu}/\mathbf{T}^{\nu-\gamma}$, which is exactly the right-hand side of equation (B-15). Thus, the same basic practical processor results by making a power series approximation here to ${}_2F_1$ instead of to the exponential in equation (B-7). The additional discussion and manipulations following equation (B-15) apply directly here as well, and equation (B-18) is the recommended final processor form, where power-law ν should be chosen for the best performance at whatever N, $\underline{M}$, and signal and noise levels are of interest.

APPENDIX C — ANALYTIC DERIVATIONS OF FALSE ALARM PROBABILITY

NORMALIZER (14) FOR $\nu = \infty$, $\underline{S} = 0$

It is presumed that random variables $\{x_n\}$ are independent and identically distributed with common probability density function $\exp(-u)$ for $u > 0$. Order these N random variables such that

$$x_1' \geq x_2' \geq \cdots \geq x_N' \; . \qquad \text{(C-1)}$$

The joint characteristic function of the latter random variables is (reference 2, equations (B-18) and (B-13))

$$f(\xi_1, \ldots, \xi_N) \equiv \overline{\exp(i\xi_1 x_1' + \cdots + i\xi_N x_N')} =$$

$$= \prod_{n=1}^{N} \frac{1}{1 - i(\xi_1 + \cdots + \xi_n)/n} \; . \qquad \text{(C-2)}$$

Now, consider normalizer (14) with $\nu = \infty$ and denote the ratio by r. Then,

$$r = \frac{X(\infty,N)}{X(1,N)} = \frac{\max(x_1, \ldots, x_N)}{(x_1 + \cdots + x_N)/N} = \frac{N\, x_1'}{x_1' + \cdots + x_N'} \; . \qquad \text{(C-3)}$$

Observe that $1 \leq r \leq N$. The exceedance distribution function of r is

$$E_r(v) = \text{Prob}(r > v) = \text{Prob}(q > 0) \; , \qquad \text{(C-4)}$$

where auxiliary random variable q is defined as

$$q \equiv N\, x_1' - v(x_1' + \cdots + x_N') = (N-v)\, x_1' - v(x_2' + \cdots + x_N') \; . \qquad \text{(C-5)}$$

The characteristic function of random variable $\mathbf{q}$ is

$$f_q(\xi) \equiv \overline{\exp(i\xi\mathbf{q})} = \overline{\exp\{i\xi(N - v)\mathbf{x}_1' - i\xi v(\mathbf{x}_2' + \cdots + \mathbf{x}_N')\}} =$$

$$= f(\xi(N - v), -\xi v, \ldots, -\xi v) = \frac{1}{\prod_{n=1}^{N}\left(1 - i\xi\,\frac{N - nv}{n}\right)} . \tag{C-6}$$

The mean of $\mathbf{q}$ follows immediately as

$$\overline{\mathbf{q}} = \sum_{n=1}^{N} \frac{N - nv}{n} = N\left(\sum_{n=1}^{N} \frac{1}{n} - v\right) . \tag{C-7}$$

The exceedance distribution function of $\mathbf{r}$ is (reference 9, equation (5))

$$E_r(v) = \text{Prob}(\mathbf{q} > 0) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{d\xi}{\xi}\,\text{Im}\{f_q(\xi)\} = \frac{1}{i2\pi}\int_C \frac{d\xi}{\xi}\,f_q(\xi) , \tag{C-8}$$

where contour C is the real ξ-axis, except for a downward detour at the origin. The function $f_q(\xi)$ in equation (C-6) has poles at

$$\xi_n = i\,\frac{n}{nv - N} \quad \text{for } 1 \le n \le N . \tag{C-9}$$

Let integer $K = \text{INT}(N/v)$, which is the largest integer less than or equal to N/v. Then, K of the N poles in equation (C-9) lie below the real axis, while the remainder are above the real axis. Therefore, for $1 \le v \le N$, equation (C-8) yields

$$E_r(v) = -\sum_{k=1}^{K} \text{Res}\left(\frac{f_q(\xi)}{\xi}; \xi_k\right) = \sum_{k=1}^{K} \alpha_k \left(1 - \frac{kv}{N}\right)^{N-1} , \tag{C-10}$$

where

$$\alpha_k = \prod_{\substack{n=1 \\ n \neq k}}^{N} \left(\frac{n}{n-k}\right) = (-1)^{k-1} \binom{N}{k} . \qquad \text{(C-11)}$$

Although equation (C-11) is an alternating sequence, the series in equation (C-10) decays quickly with k and has proven numerically useful, even for values of N as large as 1024. Substitution of equation (C-11) yields the final result

$$E_r(v) = \sum_{k=1}^{K} (-1)^{k-1} \binom{N}{k} \left(1 - \frac{kv}{N}\right)^{N-1} \quad \text{for } 1 \leq v \leq N . \qquad \text{(C-12)}$$

As checks on this result, equation (C-12) yields $E_r(N) = 0$ and $E_r(1) = 1$, as required.

NORMALIZER (14) FOR $\nu = 2$, $N = 2$, $\underline{S} = 0$

The square of normalizer (14) for $\nu = 2$ and $N = 2$ is given by

$$\mathbf{r} \equiv 2 \frac{\mathbf{x}_1^2 + \mathbf{x}_2^2}{(\mathbf{x}_1 + \mathbf{x}_2)^2} \gtrless w \equiv v^2 . \qquad \text{(C-13)}$$

The variable $\mathbf{r}$ can only take values in the range from 1 to 2. Let random variable $\mathbf{R} = \mathbf{x}_2/\mathbf{x}_1$. Then, $\mathbf{r}$ is greater than w when

$$2 \frac{1 + \mathbf{R}^2}{(1 + \mathbf{R})^2} > w , \quad \text{or} \quad \mathbf{R}^2 - 2\mathbf{R} \frac{w}{2-w} + 1 > 0 . \qquad \text{(C-14)}$$

This event occurs when $\mathbf{R} < R_1$ and when $\mathbf{R} > R_2$, where

$$R_1 = \frac{w - 2(w-1)^{\frac{1}{2}}}{2 - w} , \quad R_2 = \frac{w + 2(w-1)^{\frac{1}{2}}}{2 - w} . \qquad \text{(C-15)}$$

The exceedance distribution function of r is

$$\Pr(r > w) = \Pr(\mathbf{R} < R_1) + \Pr(R_2 < \mathbf{R}) = 1 - \Pr(\mathbf{R} > R_1) + \Pr(\mathbf{R} > R_2). \tag{C-16}$$

But since

$$\Pr(\mathbf{R} > u) = \Pr(\mathbf{x}_2 > \mathbf{x}_1 u) = \int_0^\infty db\ e^{-b} \int_{bu}^\infty da\ e^{-a} = \frac{1}{1 + u} \quad \text{for } u \geq 0 \ , \tag{C-17}$$

it follows that

$$E_r(w) = 1 - \frac{1}{1 + R_1} + \frac{1}{1 + R_2} =$$

$$= 1 - \frac{1 + (w - 1)^{\frac{1}{2}}}{2} + \frac{1 - (w - 1)^{\frac{1}{2}}}{2} = 1 - (w - 1)^{\frac{1}{2}} \tag{C-18}$$

for $1 \leq w \leq 2$.

APPENDIX D — RECEIVER OPERATING CHARACTERISTICS FOR NORMALIZER (14)

The normalizer of interest here is

$$\frac{X(\nu,N)}{X(1,N)} \gtrless v ,$$

which is equation (14) from the main text. All the ROCs in this appendix are for $\mu = 1$ and search size N = 1024. The values of power-law ν range over ∞, 3, 2.5, 2, and 1, while the values of $\underline{M}$ (the number of signal bins) range over 1, 4, 16, and 64. The simulations were done for noise level $\underline{N} = 1$; therefore, the signal level $\underline{S}$ (dB) labeled on each curve can be interpreted as the signal-to-noise ratio per bin in decibels. The number of independent trials of normalizer (14) that are utilized for each ROC are indicated in each case. The abscissa and ordinate labelings on figures D-2 to D-20 are identical to those indicated on figure D-1.

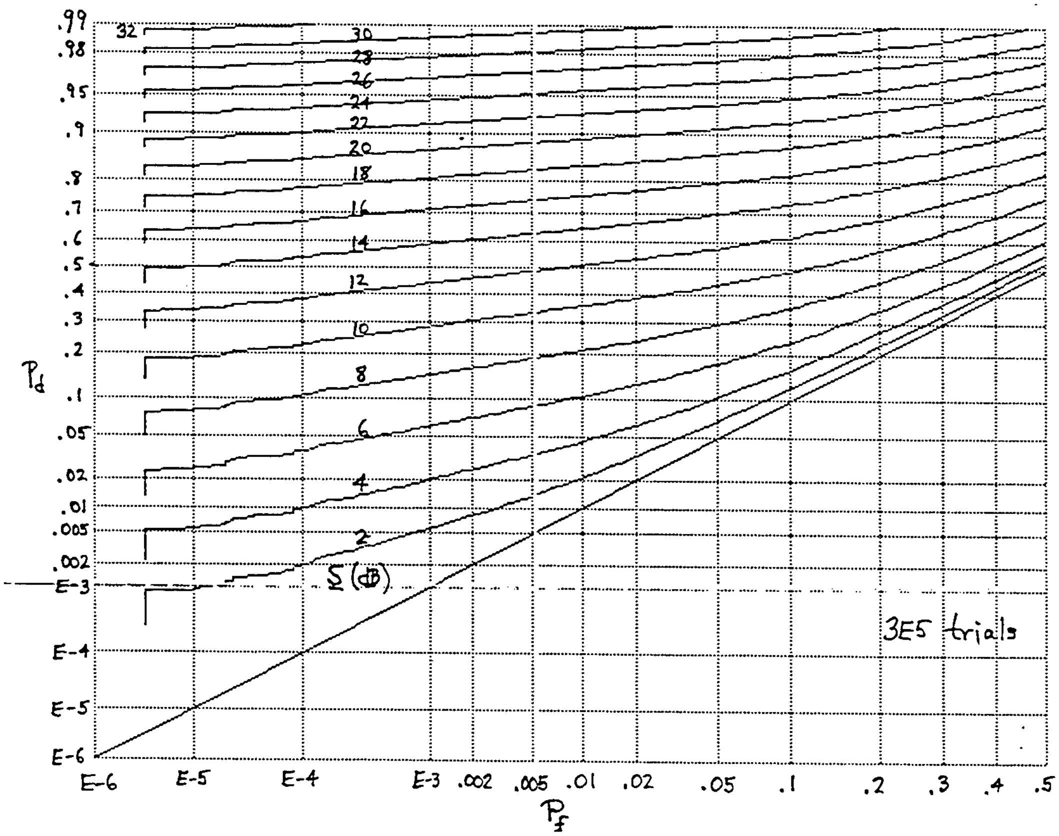

Figure D-1. ROC for M = 1, ν = ∞, μ = 1, N = 1024

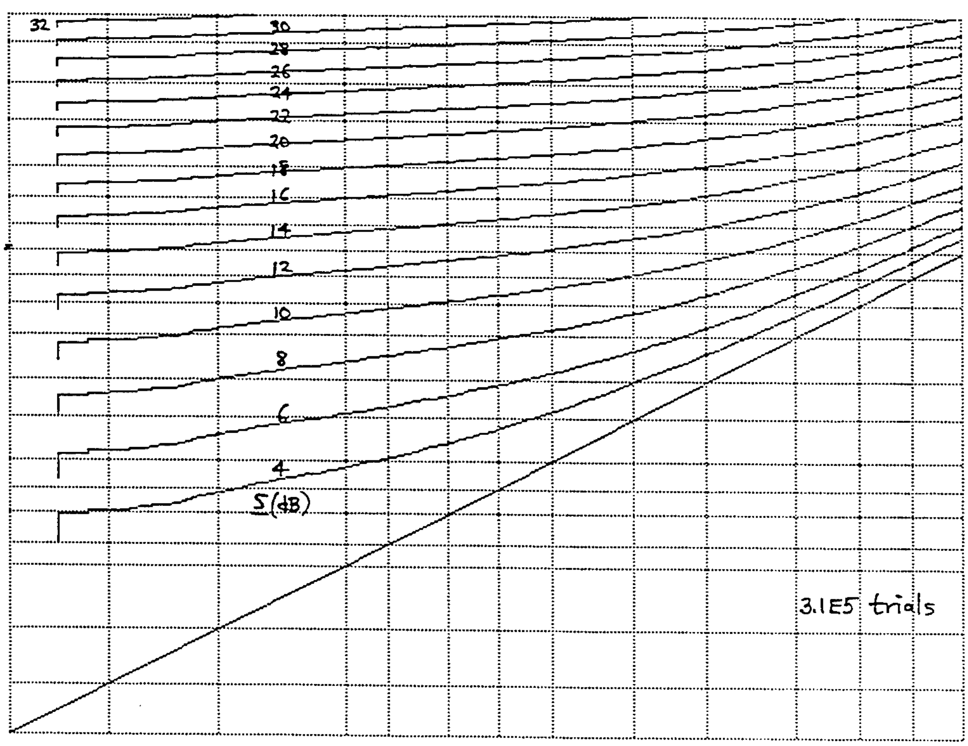

Figure D-2. ROC for M = 1, ν = 3, μ = 1, N = 1024

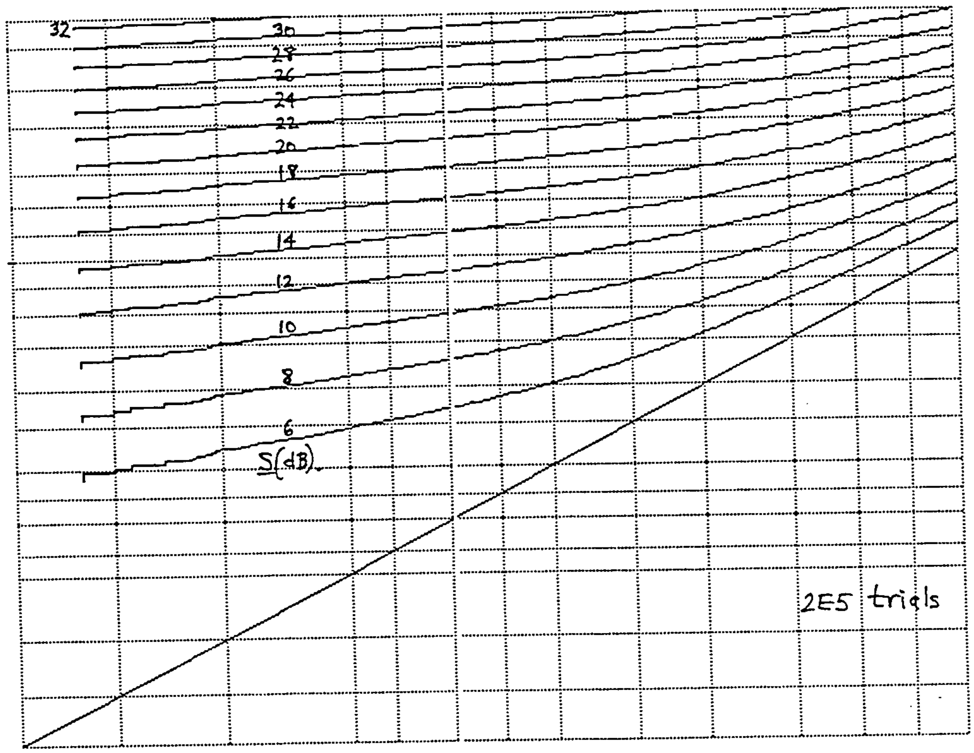

Figure D-3. ROC for $\underline{M}$ = 1, ν = 2.5, μ = 1, N = 1024

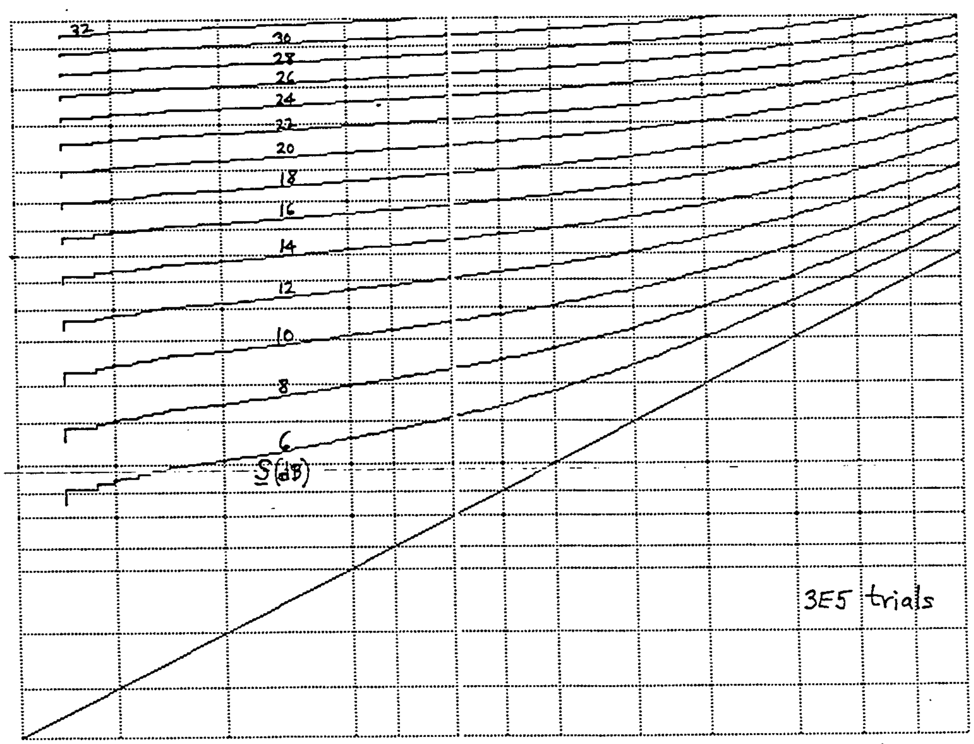

Figure D-4. ROC for $\underline{M}$ = 1, ν = 2, μ = 1, N = 1024

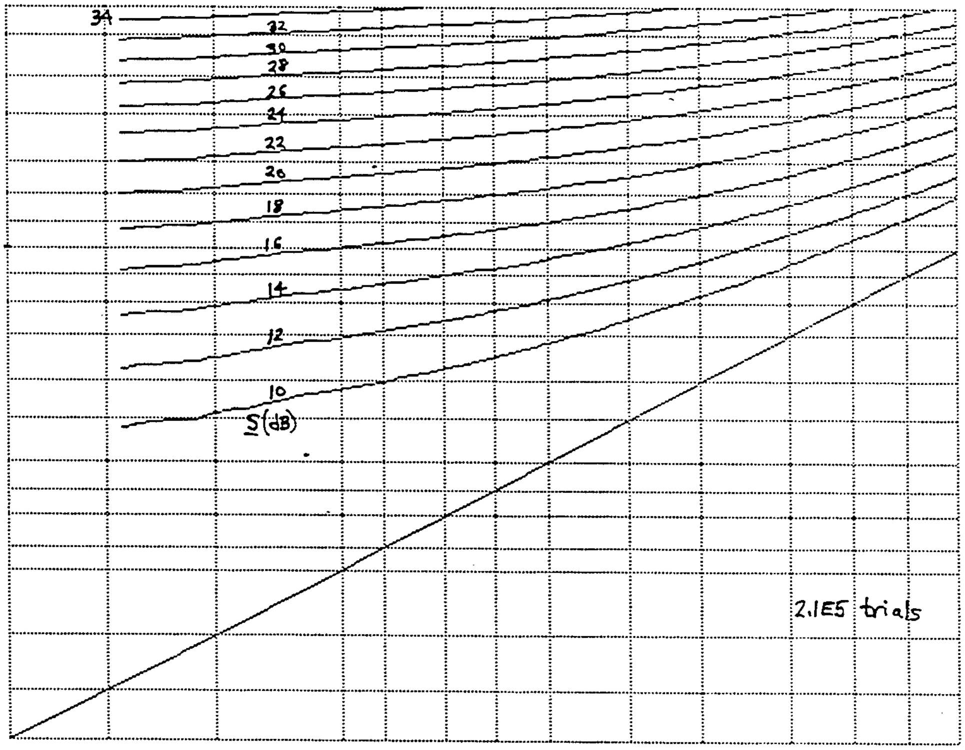

Figure D-5. ROC for $\underline{M} = 1$, $\nu = 1$, $\mu = 1$, $N = 1024$

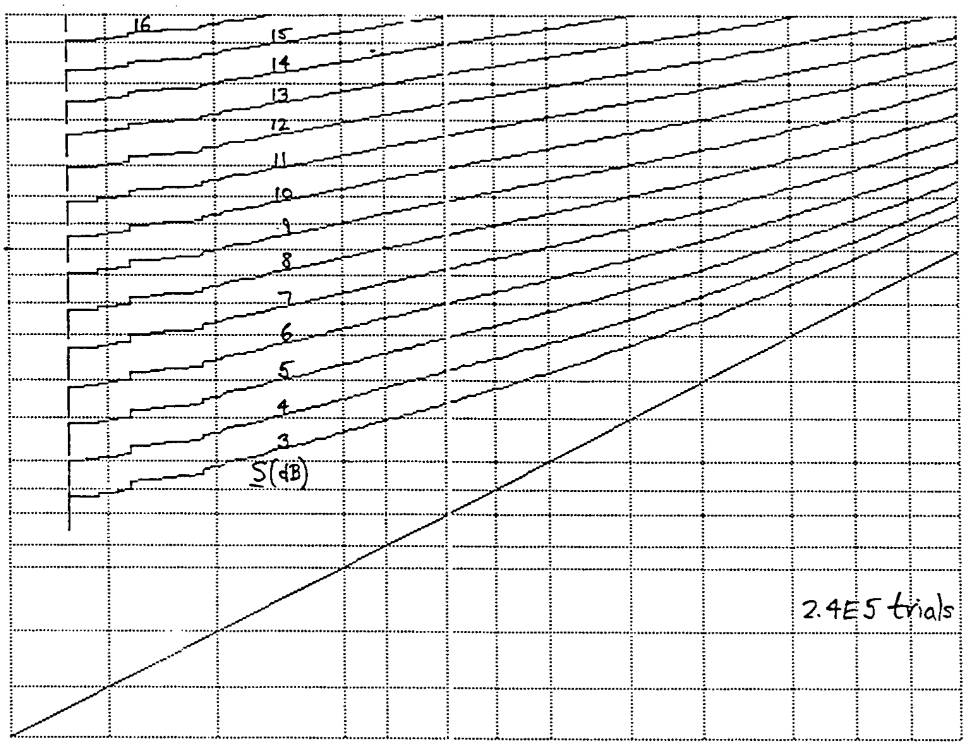

Figure D-6. ROC for $\underline{M} = 4$, $\nu = \infty$, $\mu = 1$, $N = 1024$

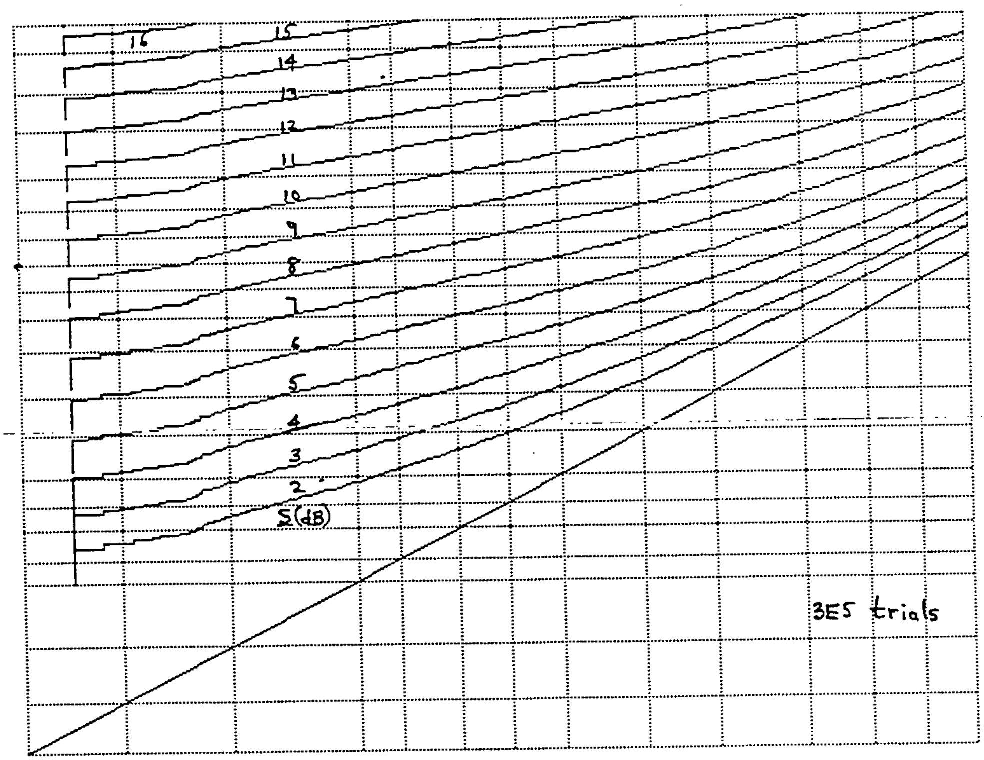

Figure D-7. ROC for M = 4, ν = 3, μ = 1, N = 1024

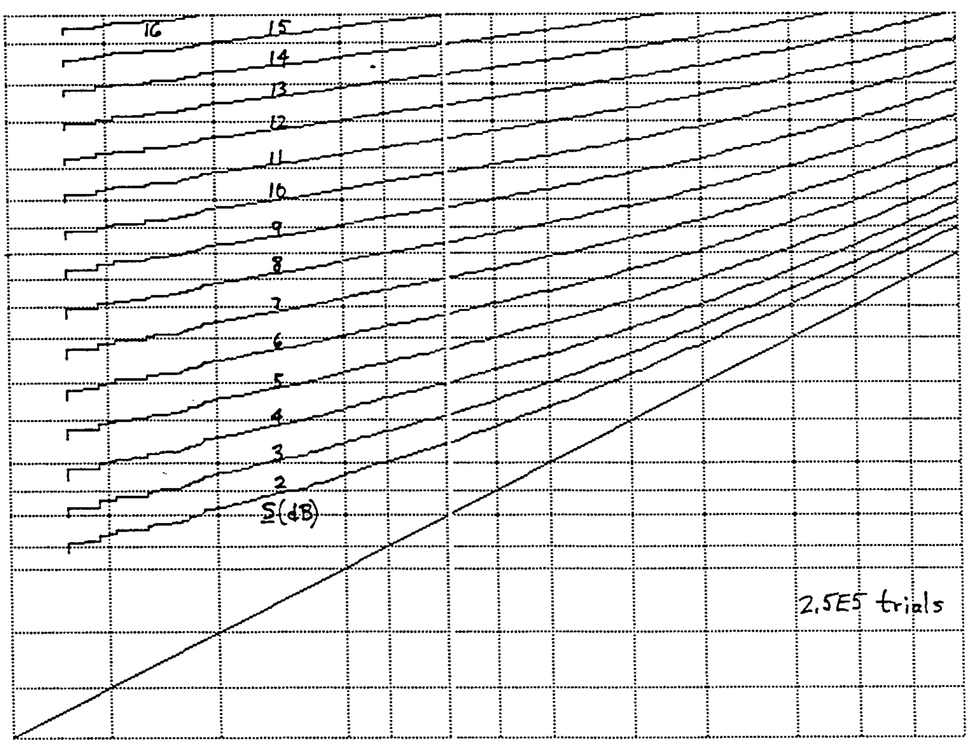

Figure D-8. ROC for M = 4, ν = 2.5, μ = 1, N = 1024

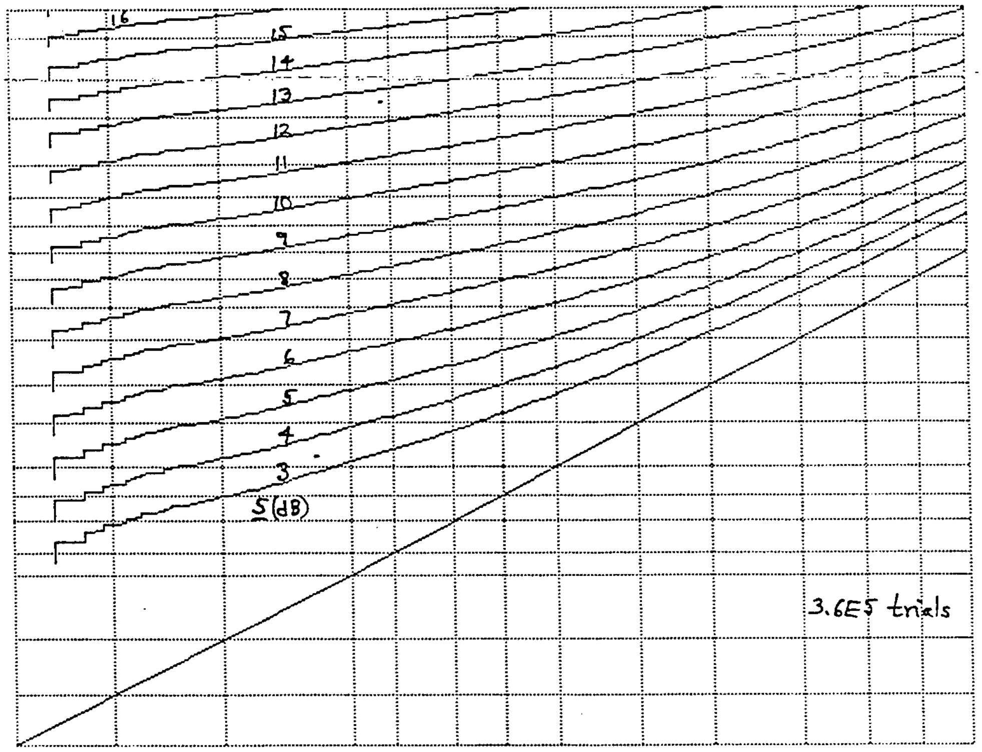

Figure D-9. ROC for M = 4, ν = 2, μ = 1, N = 1024

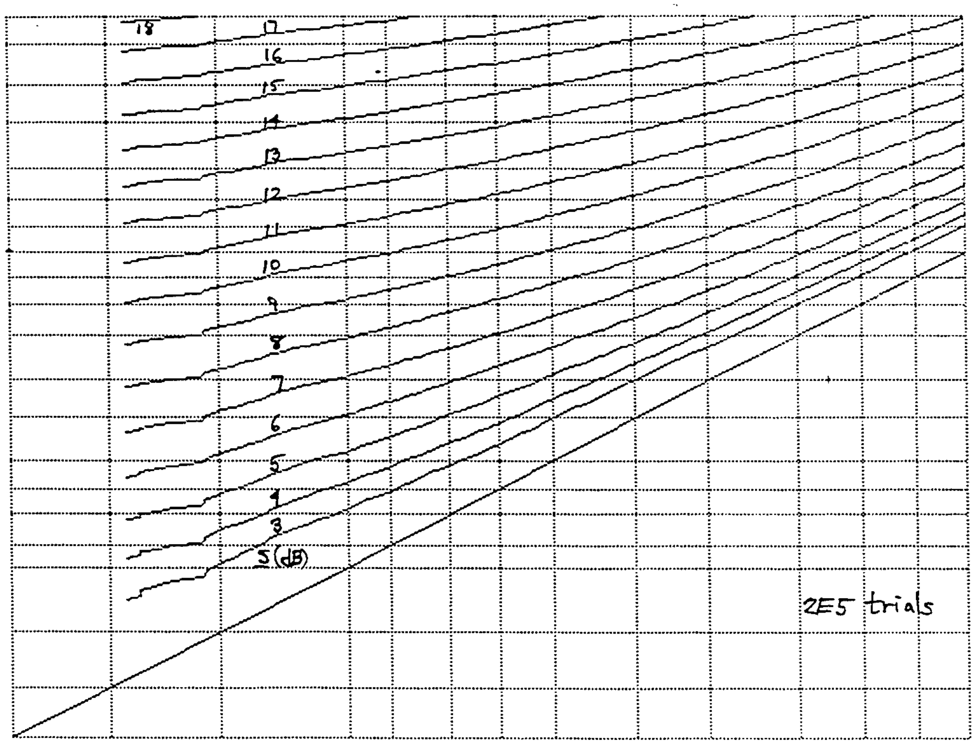

Figure D-10. ROC for M = 4, ν = 1, μ = 1, N = 1024

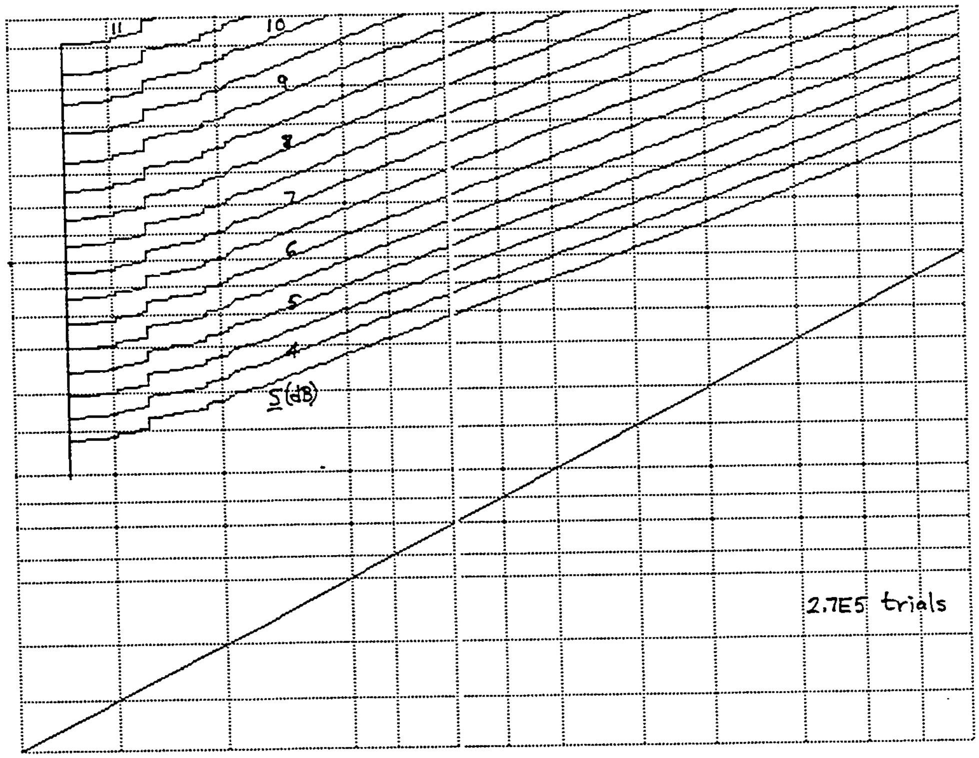

Figure D-11. ROC for $\underline{M}$ = 16, $\nu = \infty$, μ = 1, N = 1024

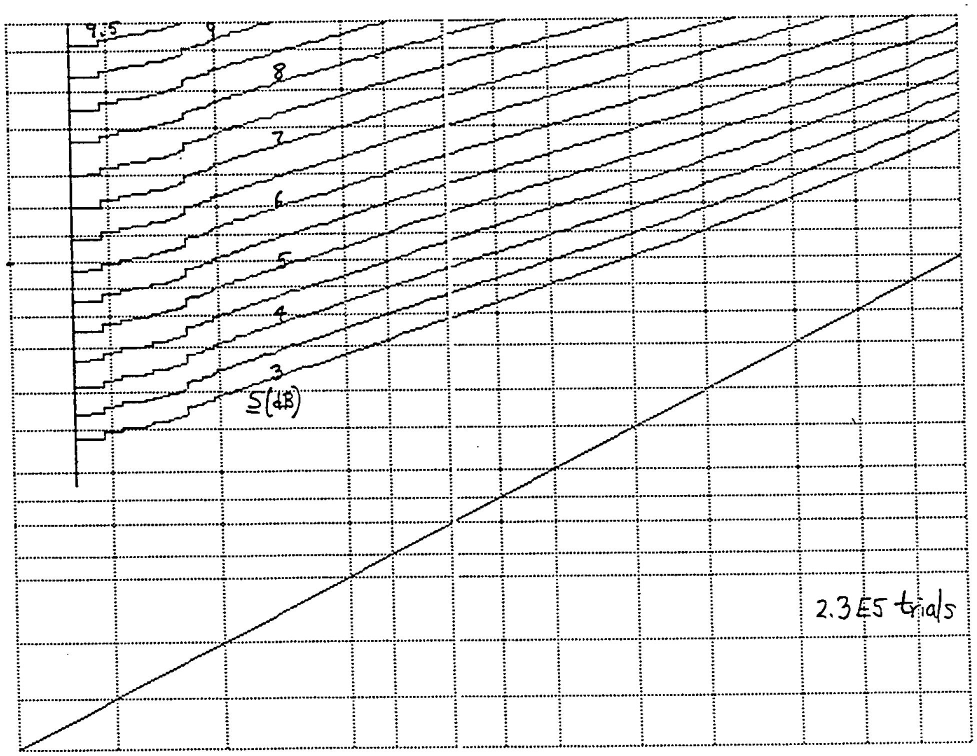

Figure D-12. ROC for $\underline{M}$ = 16, ν = 3, μ = 1, N = 1024

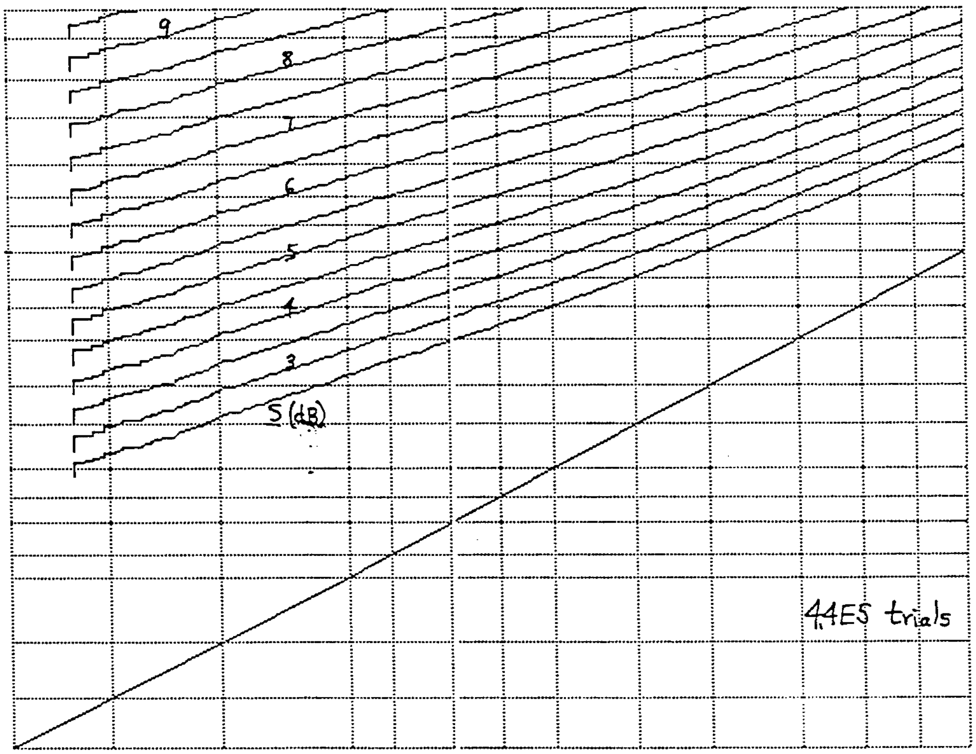

Figure D-13. ROC for $\underline{M}$ = 16, ν = 2.5, μ = 1, N = 1024

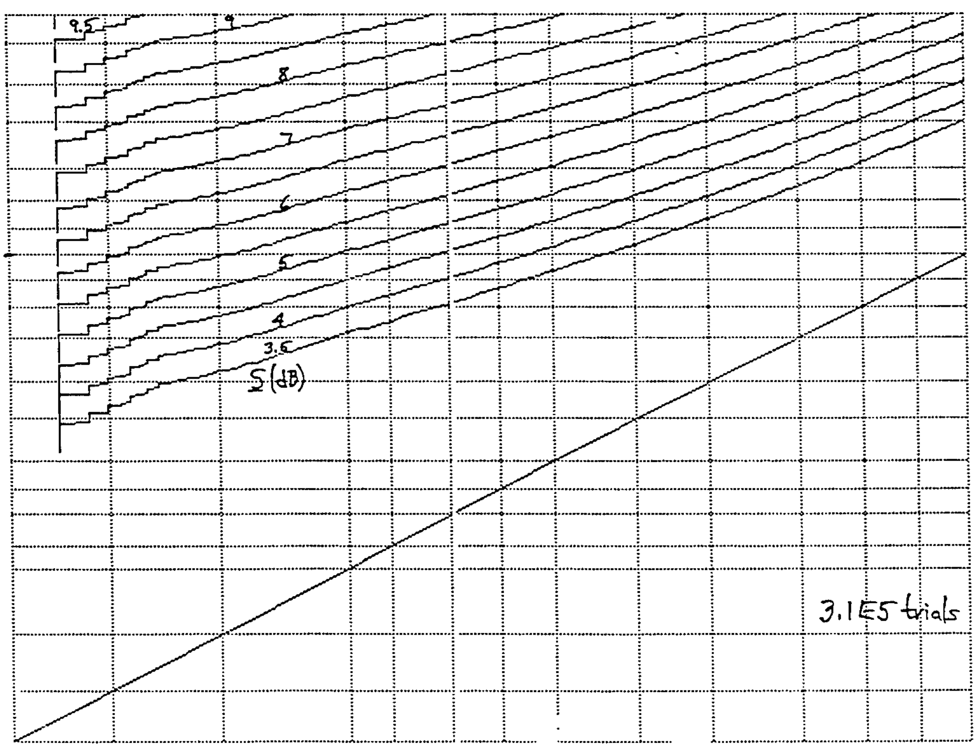

Figure D-14. ROC for $\underline{M}$ = 16, ν = 2, μ = 1, N = 1024

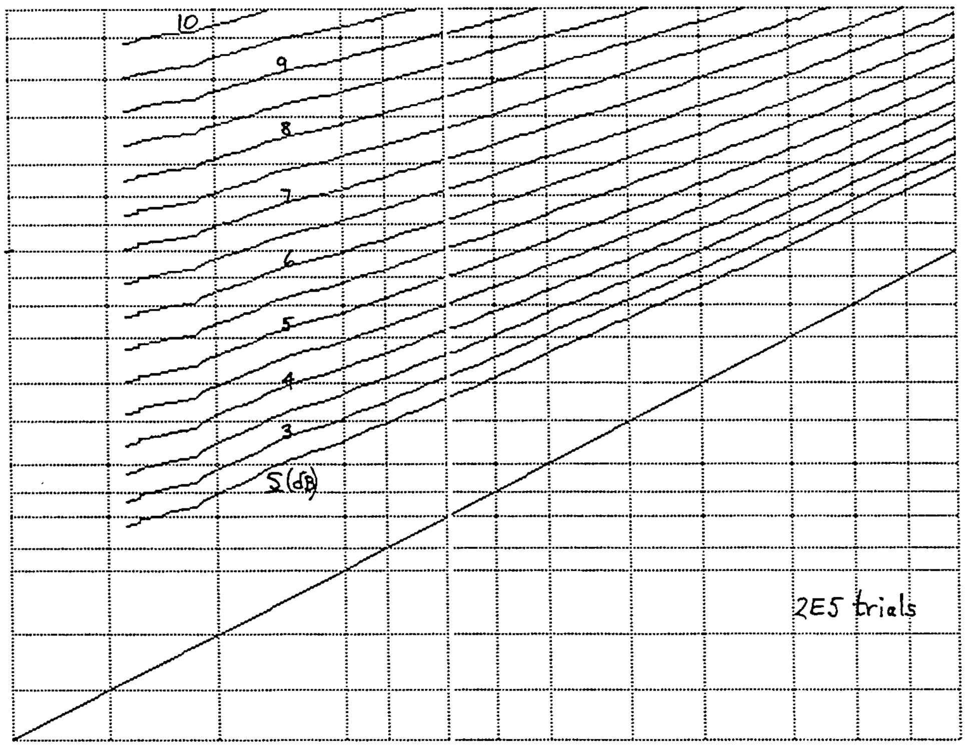

Figure D-15. ROC for $\underline{M}$ = 16, ν = 1, μ = 1, N = 1024

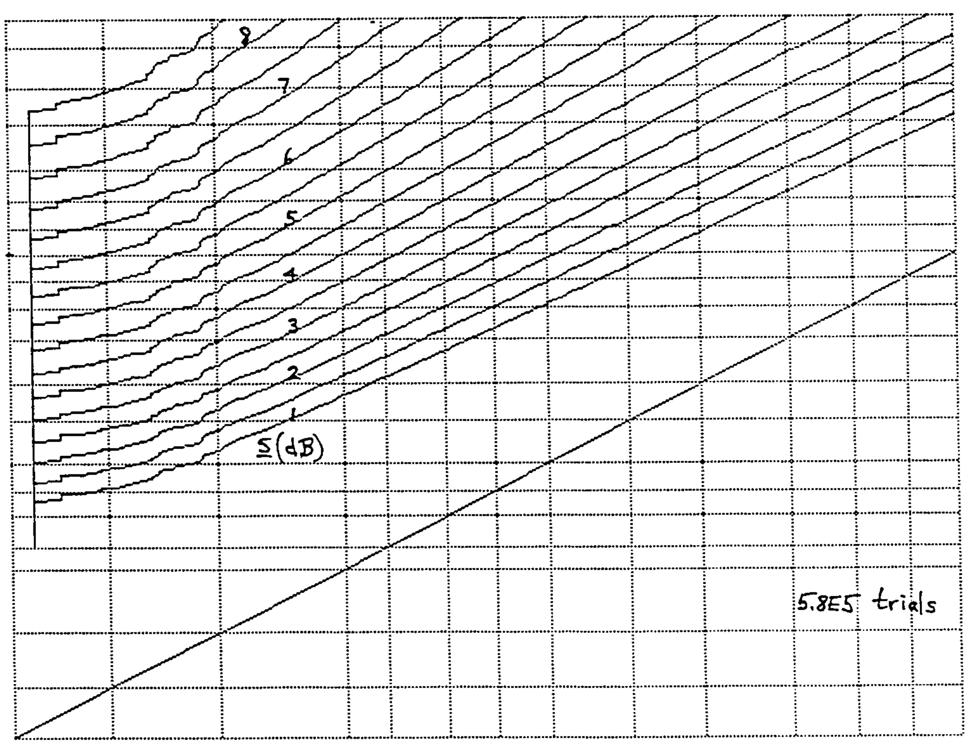

Figure D-16. ROC for $\underline{M}$ = 64, ν = ∞, μ = 1, N = 1024

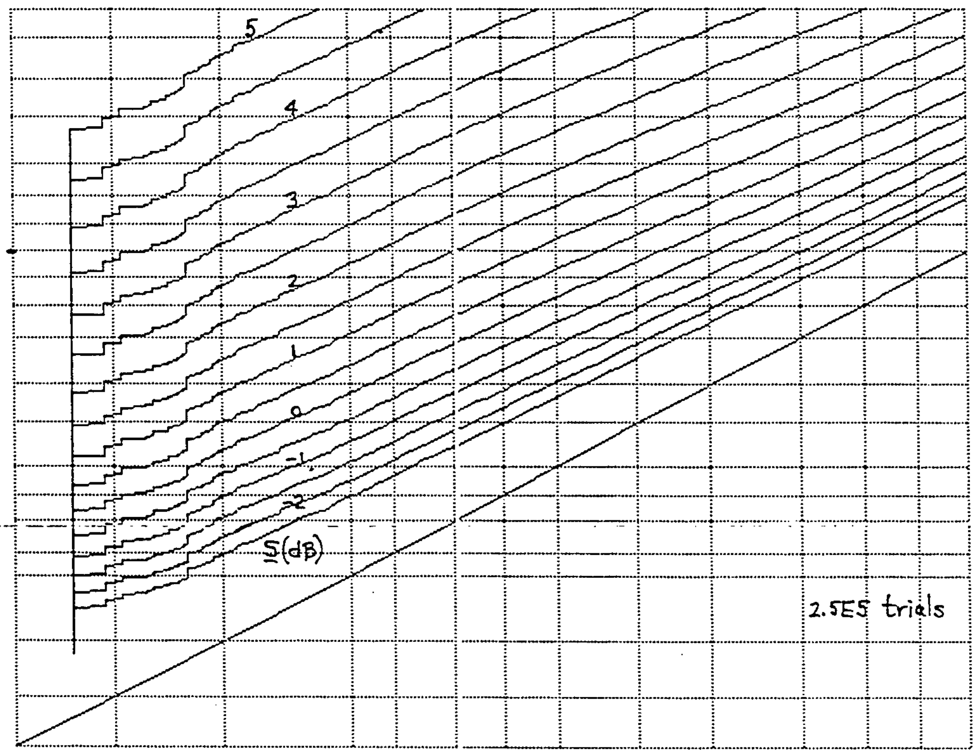

Figure D-17. ROC for $\underline{M}$ = 64, ν = 3, μ = 1, N = 1024

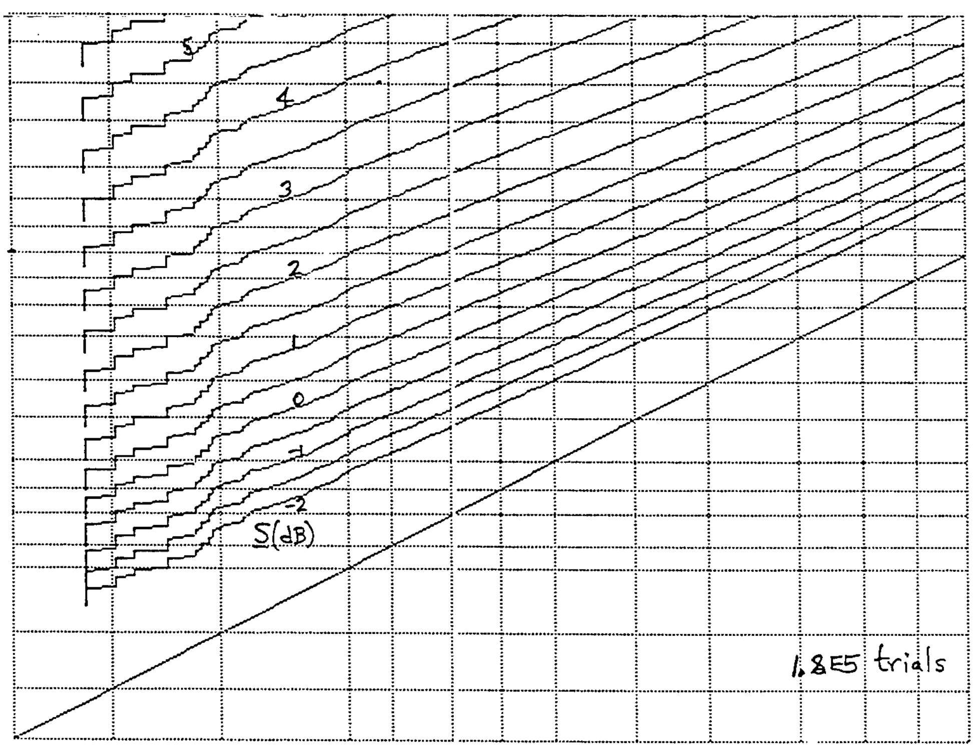

Figure D-18. ROC for $\underline{M}$ = 64, ν = 2.5, μ = 1, N = 1024

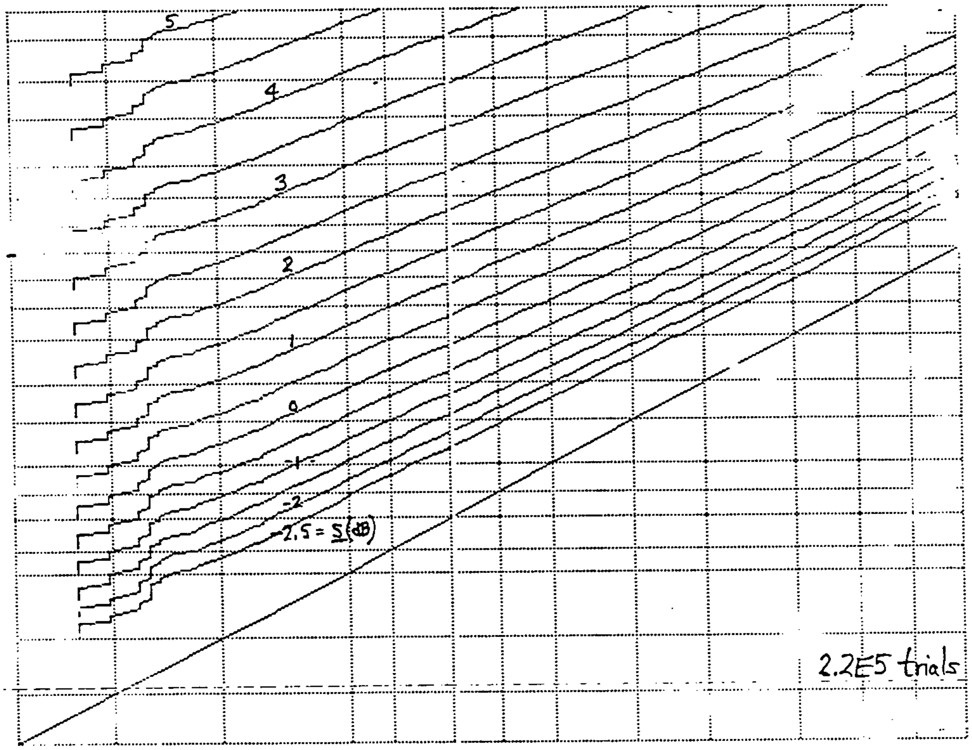

Figure D-19. ROC for M = 64, ν = 2, μ = 1, N = 1024

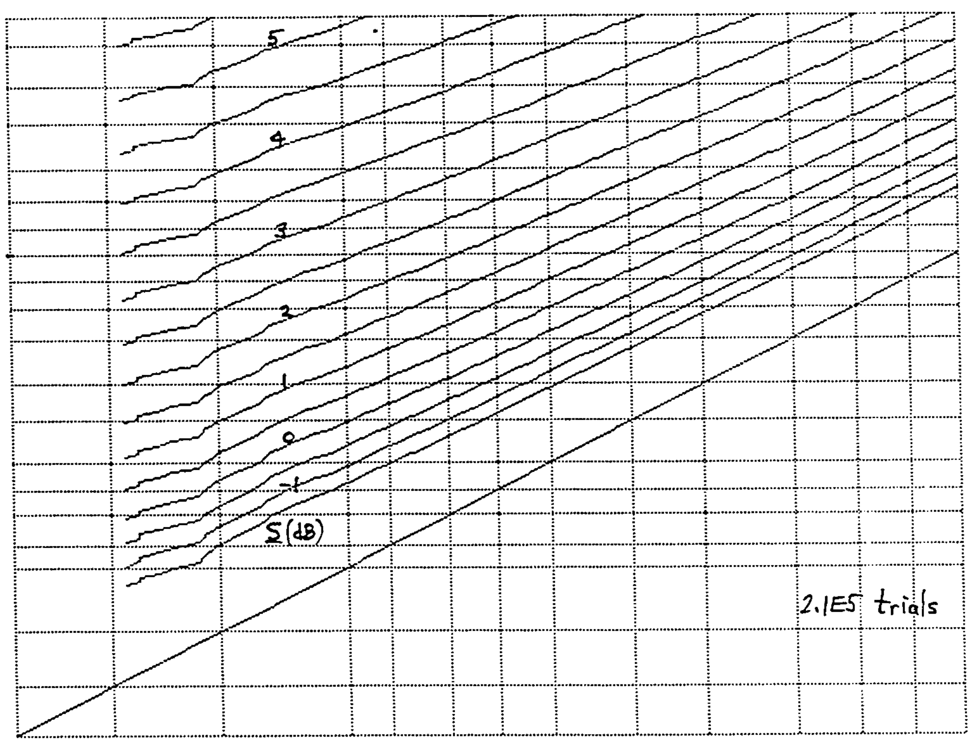

Figure D-20. ROC for M = 64, ν = 1, μ = 1, N = 1024

APPENDIX E — LIMITING NORMALIZER AS $\mu \to \nu$

The general self-reference normalizer is given by equations (13) and (7) as

$$\frac{X(\nu,N)}{X(\mu,N)} = \frac{\left(\frac{1}{N}\sum_{n=1}^{N} x_n^{\nu}\right)^{1/\nu}}{\left(\frac{1}{N}\sum_{n=1}^{N} x_n^{\mu}\right)^{1/\mu}} . \tag{E-1}$$

Let $\mu = \nu - \varepsilon$, where $\varepsilon > 0$. Then, an expansion of equation (E-1) yields

$$\frac{X(\nu,N)}{X(\mu,N)} \sim 1 + \frac{\varepsilon}{\nu^2}\left(\frac{L}{P} - \ln(P)\right) \quad \text{as } \varepsilon \to 0 , \tag{E-2}$$

where

$$L \equiv \sum_{n=1}^{N} x_n^{\nu} \ln(x_n^{\nu}) , \quad P \equiv \sum_{n=1}^{N} x_n^{\nu} . \tag{E-3}$$

Therefore, the dominant data-dependent term in equation (E-2) is

$$\frac{L}{P} - \ln(P) = \sum_{n=1}^{N} \frac{x_n^{\nu}}{P} \ln\left(\frac{x_n^{\nu}}{P}\right) . \tag{E-4}$$

This processor obviously has a constant false alarm probability capability, regardless of the value of ν.

However, since $0 \le x_n^{\nu} < P$, every term in the sum is negative. The most negative value that the sum in equation (E-4) can take on is $-\ln(N)$, when all the x_n are equal. Therefore, the processor that is adopted becomes

$$\ln(N) + \frac{L}{P} - \ln(P) = \frac{1}{N} \sum_{n=1}^{N} \frac{x_n^{\nu}}{P/N} \ln\left(\frac{x_n^{\nu}}{P/N}\right) , \tag{E-5}$$

which always has positive outputs. The most positive value of equation (E-5) is ln(N), attained when all x_n are zero except one. This latter processor (E-5) has a constant false alarm probability capability since replacement of $\{x_n\}$ by $\{a\ x_n\}$ yields the same output, independent of scale factor a.

APPENDIX F — RECEIVER OPERATING CHARACTERISTICS FOR NORMALIZER (16)

The normalizer of interest here is

$$\frac{X(\nu,N)}{X(\mu,N)} \gtrless v \ ,$$

which is equation (16) from the main text. All the ROCs in this appendix are for $\underline{M}$ = 256 and search size N = 1024. The values of power-law ν and averager parameter μ range over a variety of values in the neighborhood of ν = 2 and μ = 1. The simulations were done for noise level $\underline{N}$ = 1; therefore, the signal level $\underline{S}$ (dB) labeled on each curve can be interpreted as the signal-to-noise ratio per bin in decibels. The number of independent trials of normalizer (16) that are utilized for each ROC are indicated in each case. The abscissa and ordinate labelings on figures F-2 to F-30 are identical to those indicated on figure F-1. The ROCs for processor (16) are ordered in such a way that the better a particular pair (ν,μ) performs, the earlier it is listed in this appendix.

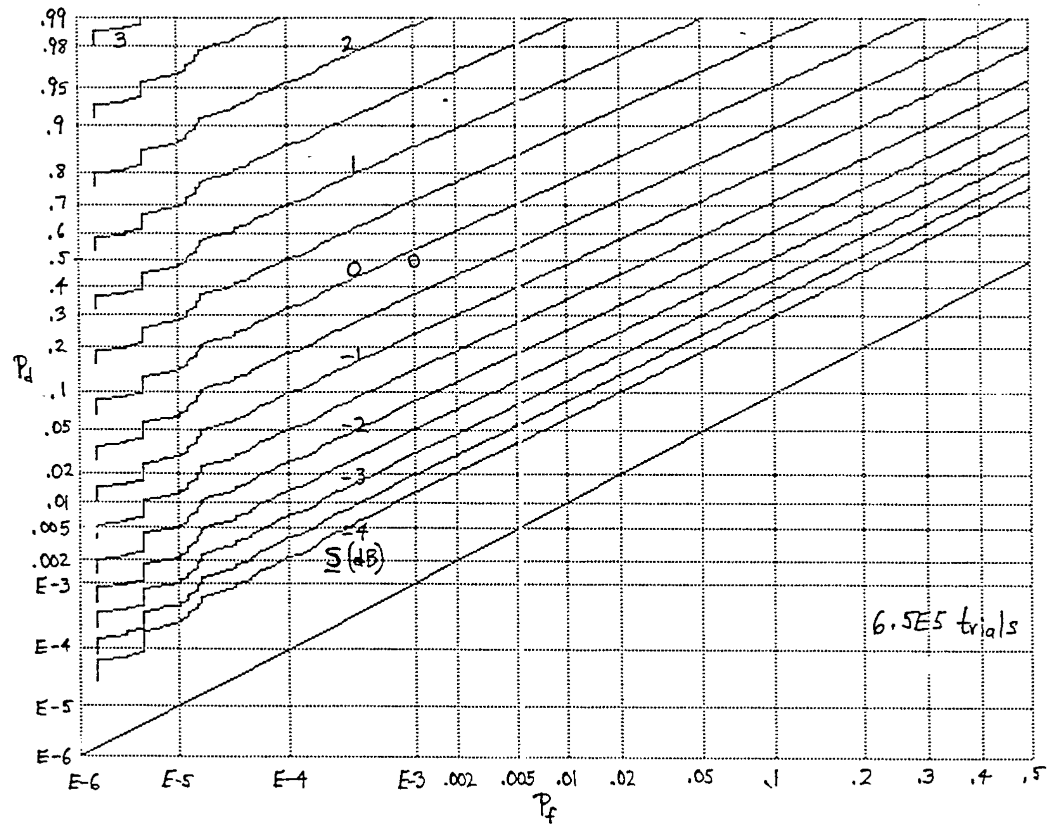

Figure F-1. ROC for $\underline{M}$ = 256, ν = 1.5, μ = 1, N = 1024

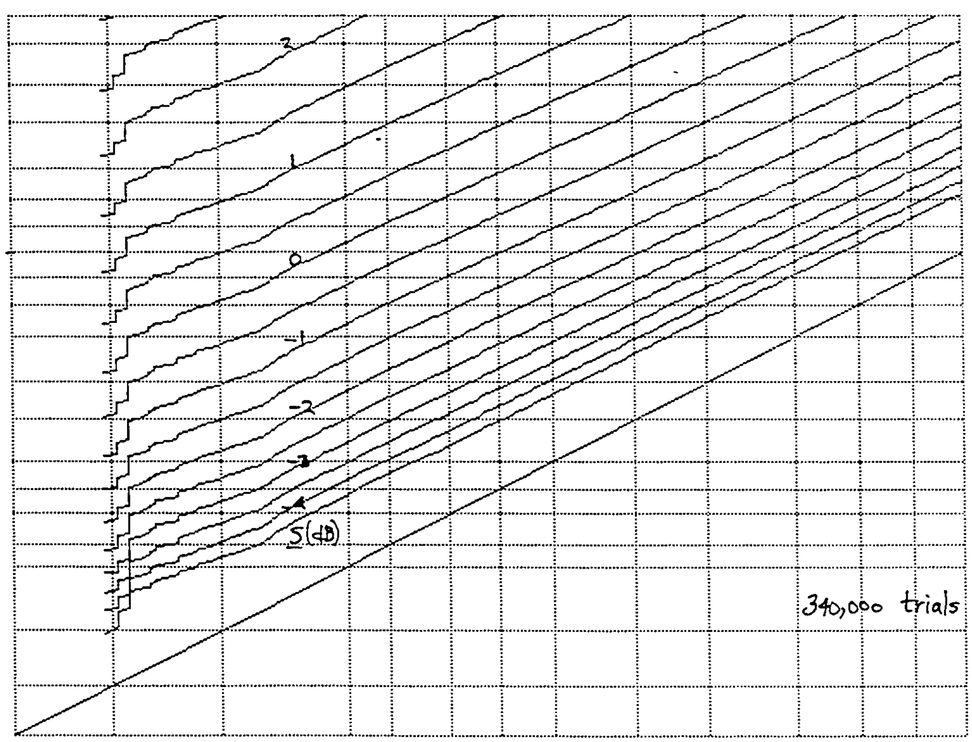

Figure F-2. ROC for $\underline{M}$ = 256, ν = 1.5, μ = 1.25, N = 1024

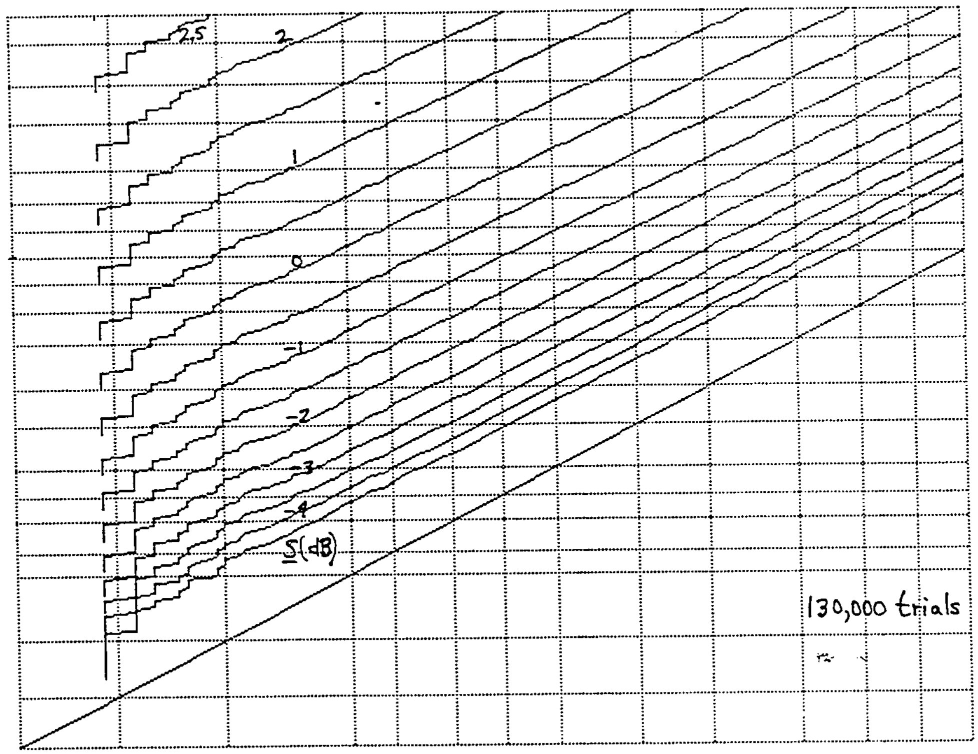

Figure F-3. ROC for M = 256, ν = 1.25, μ = 1.25, N = 1024

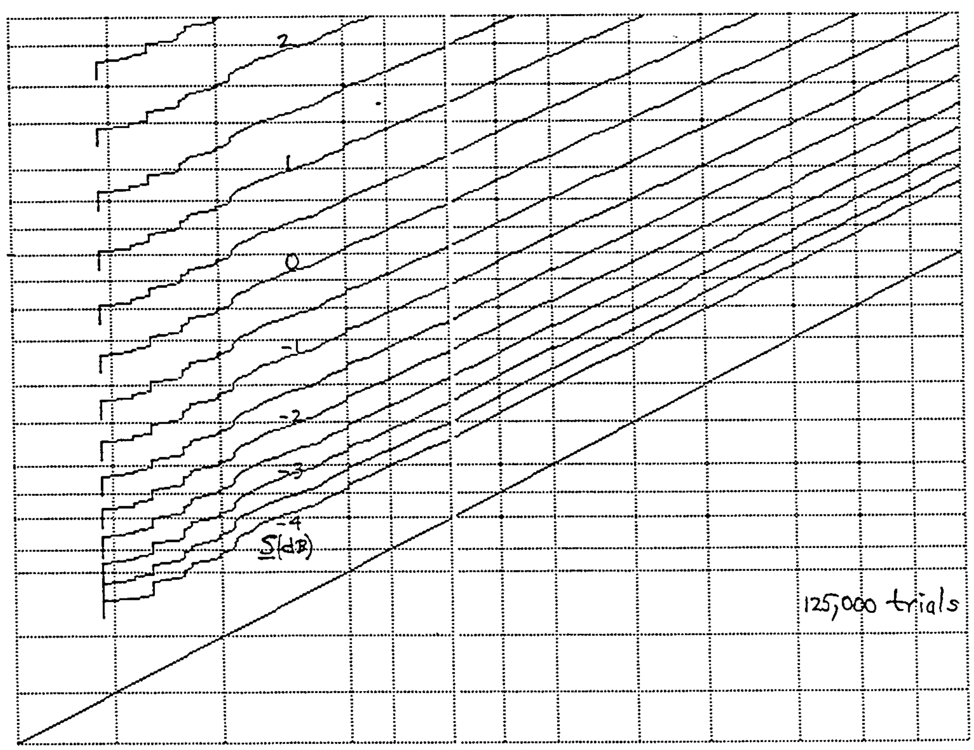

Figure F-4. ROC for M = 256, ν = 1.25, μ = 1, N = 1024

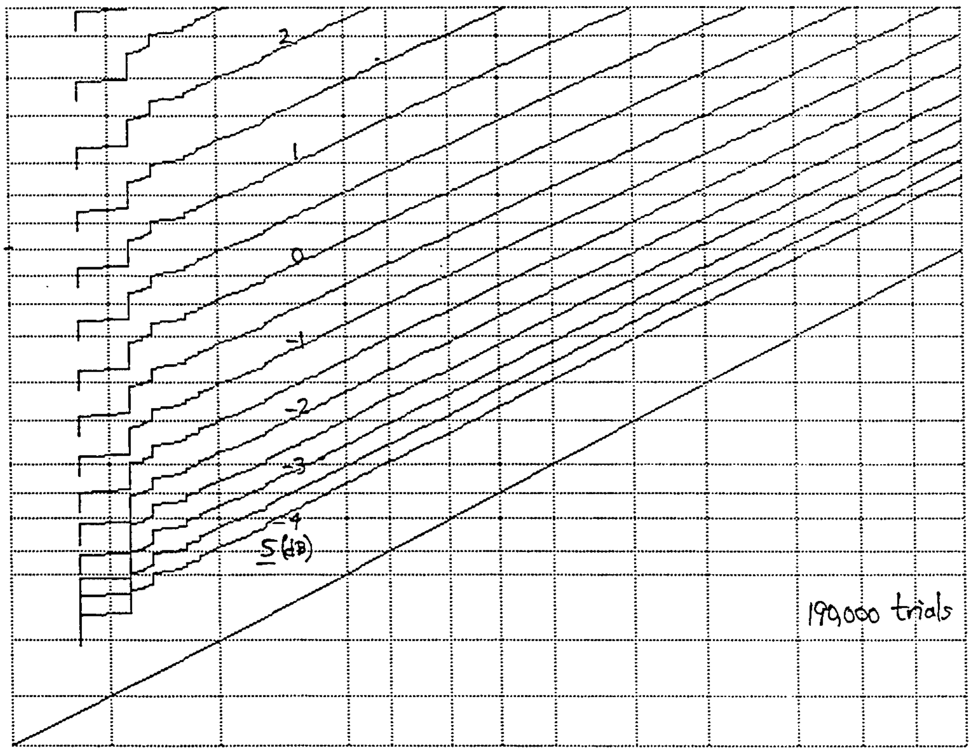

Figure F-5. ROC for M = 256, ν = 1.75, μ = 1, N = 1024

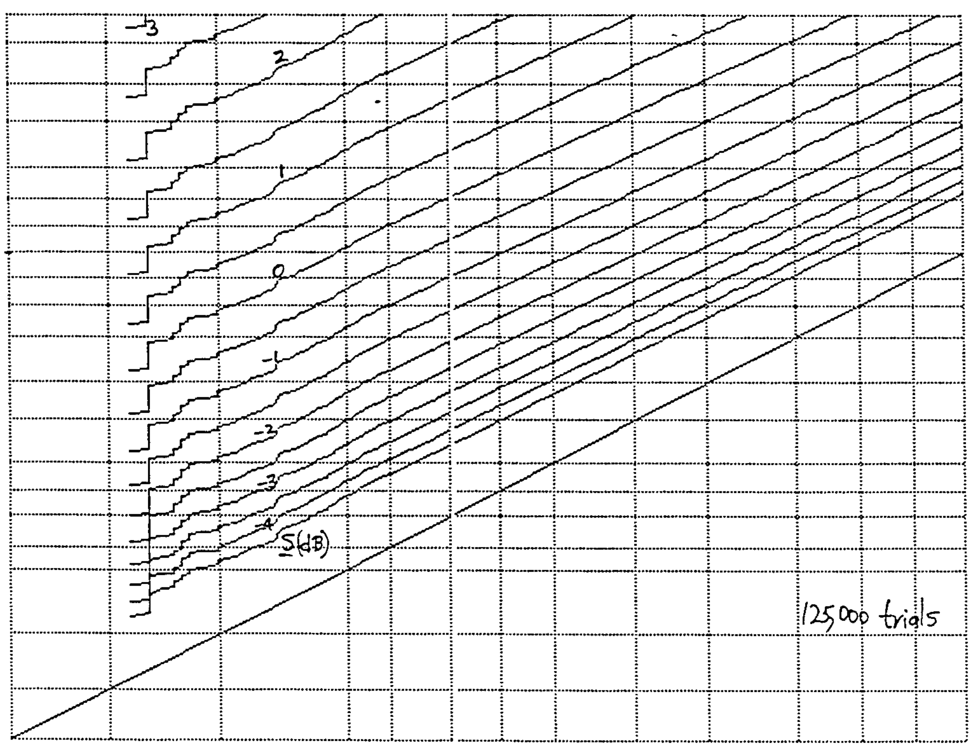

Figure F-6. ROC for M = 256, ν = 1.5, μ = 1.5, N = 1024

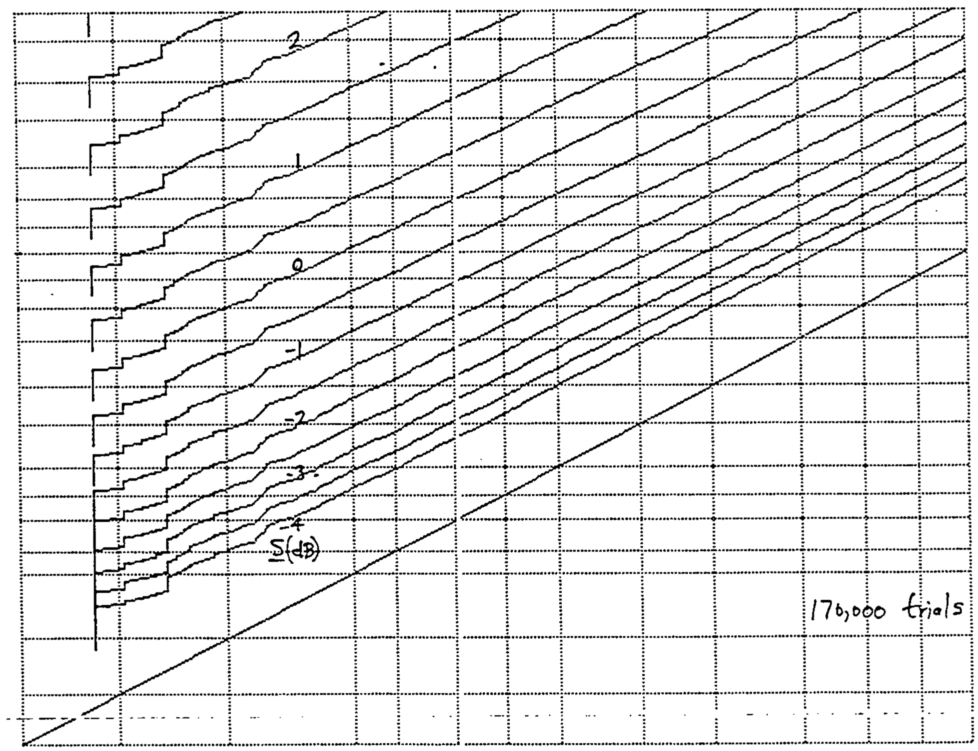

Figure F-7. ROC for M = 256, ν = 2, μ = 0.5, N = 1024

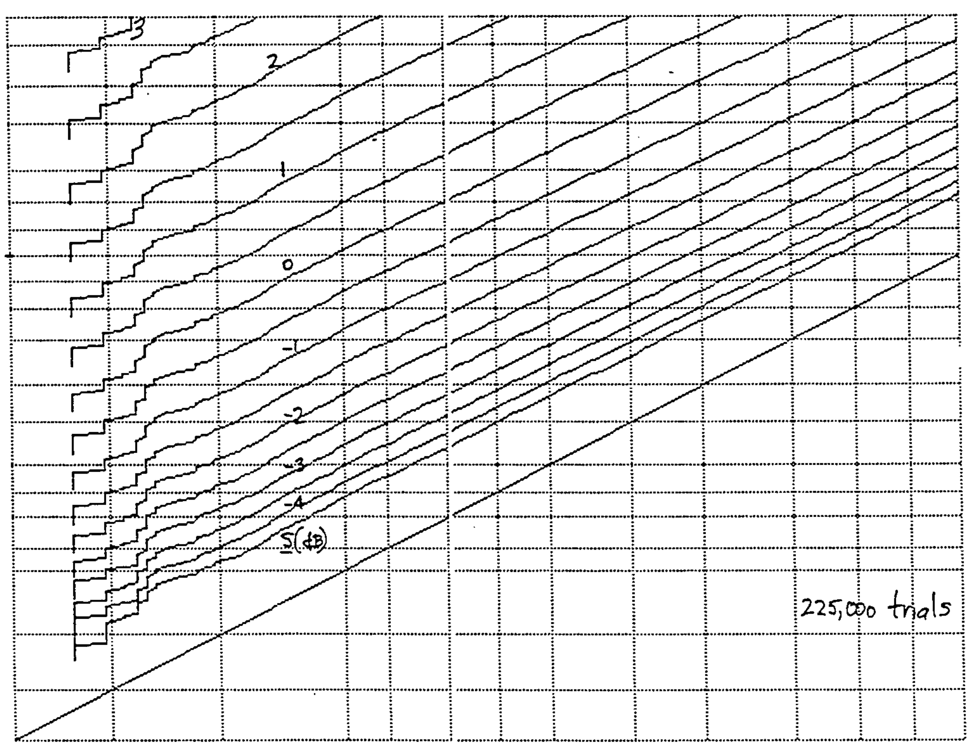

Figure F-8. ROC for M = 256, ν = 2, μ = 1, N = 1024

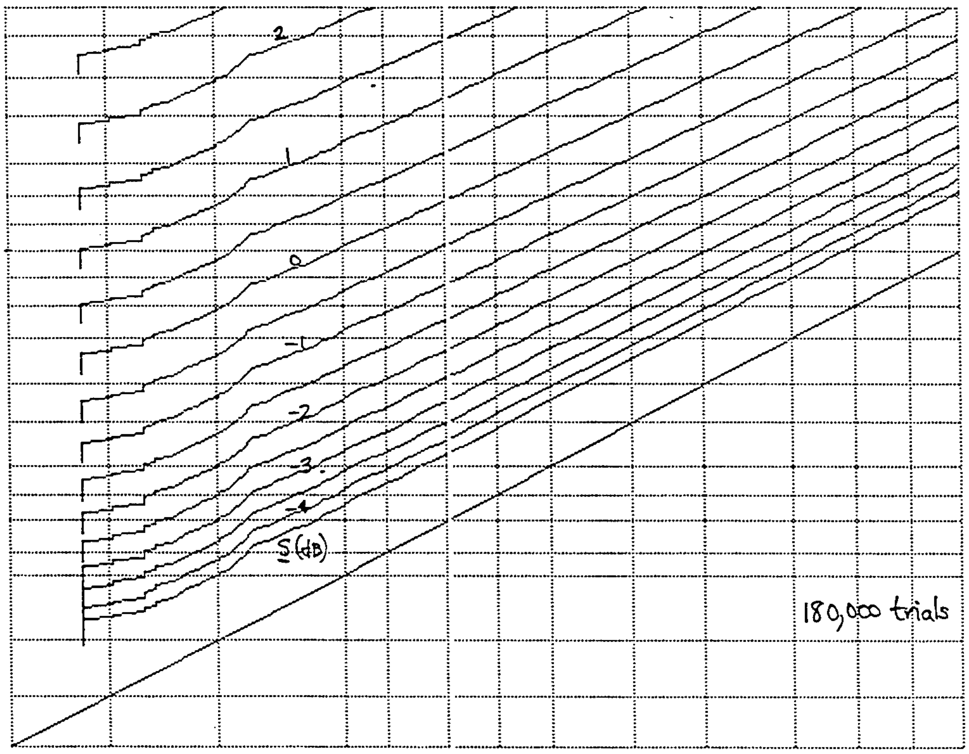

Figure F-9. ROC for $\underline{M}$ = 256, ν = 1.5, μ = 0.75, N = 1024

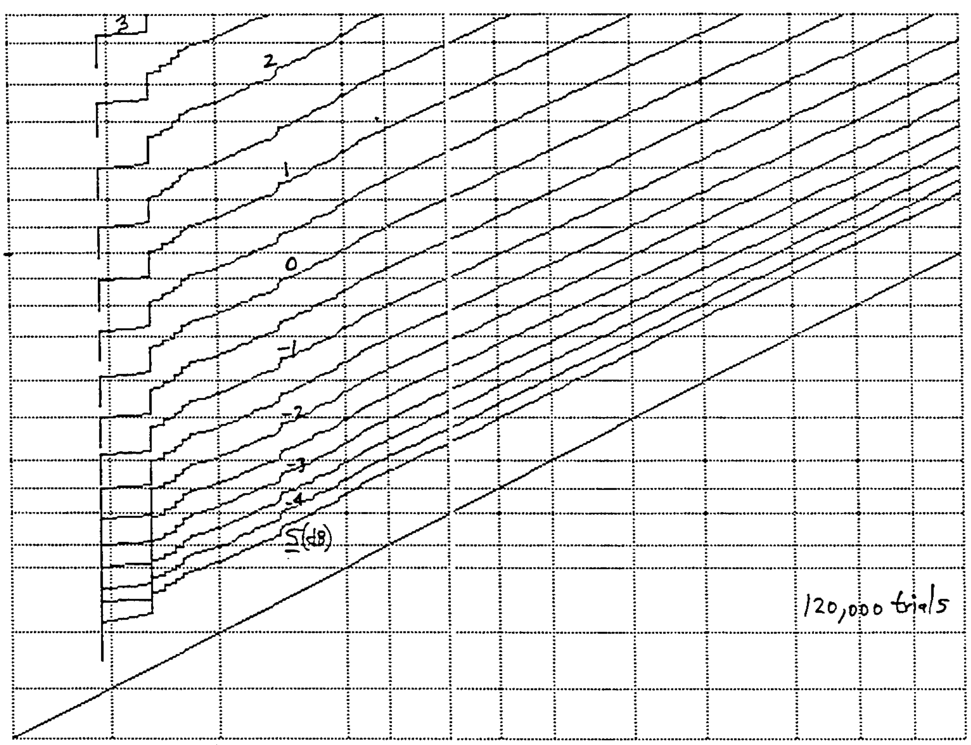

Figure F-10. ROC for $\underline{M}$ = 256, ν = 1.75, μ = 1.25, N = 1024

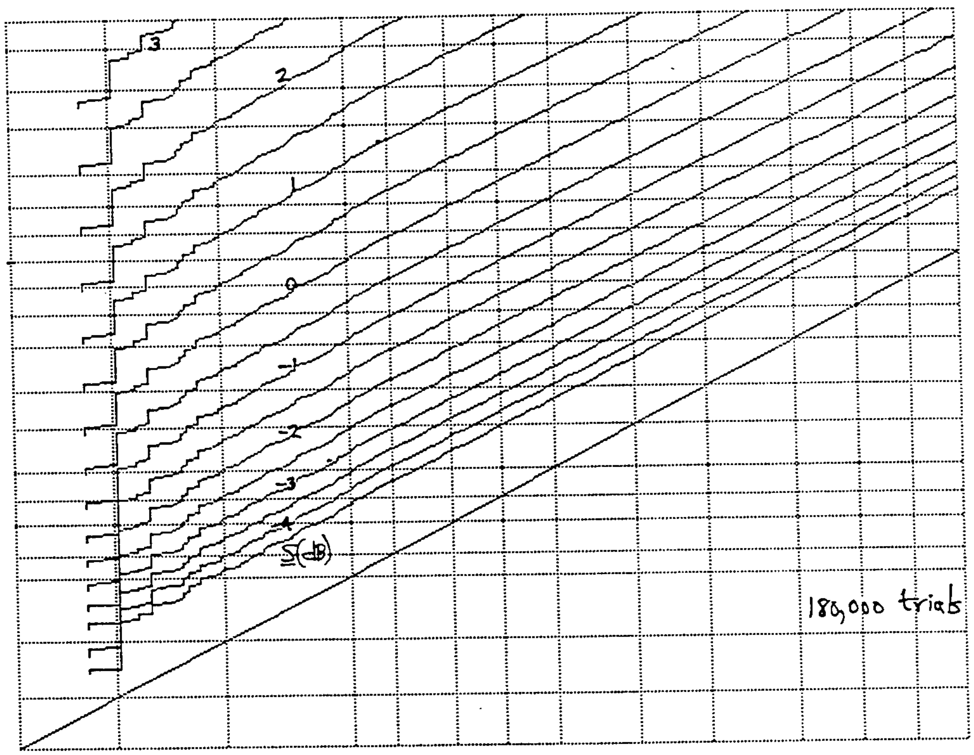

Figure F-11. ROC for $\underline{M}$ = 256, ν = 1.75, μ = 1.5, N = 1024

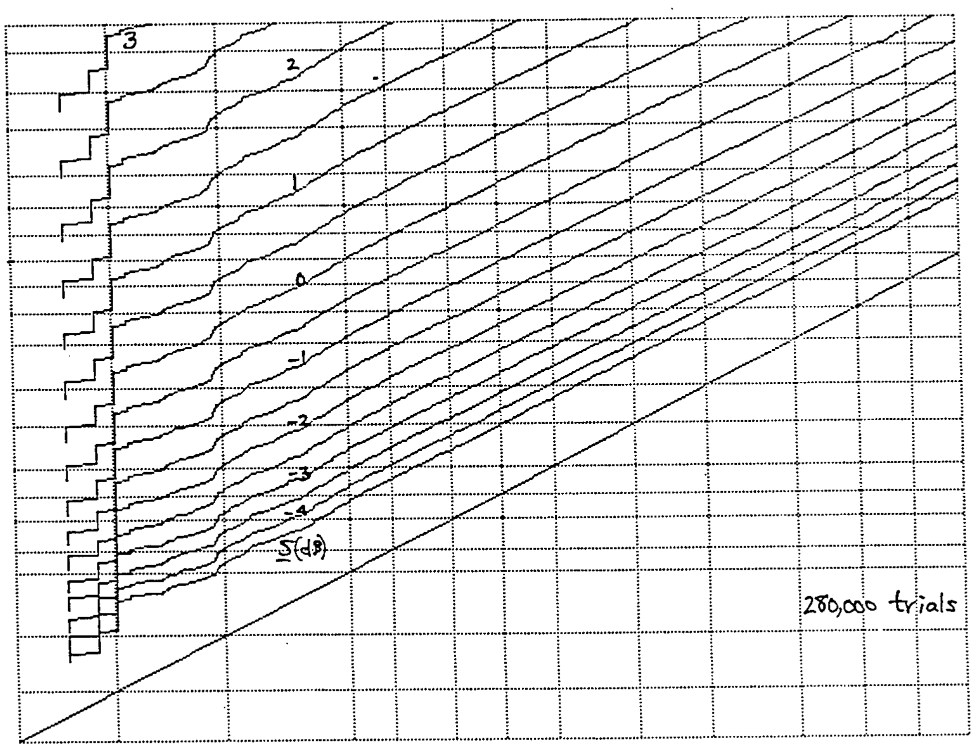

Figure F-12. ROC for $\underline{M}$ = 256, ν = 2.5, μ = 0.5, N = 1024

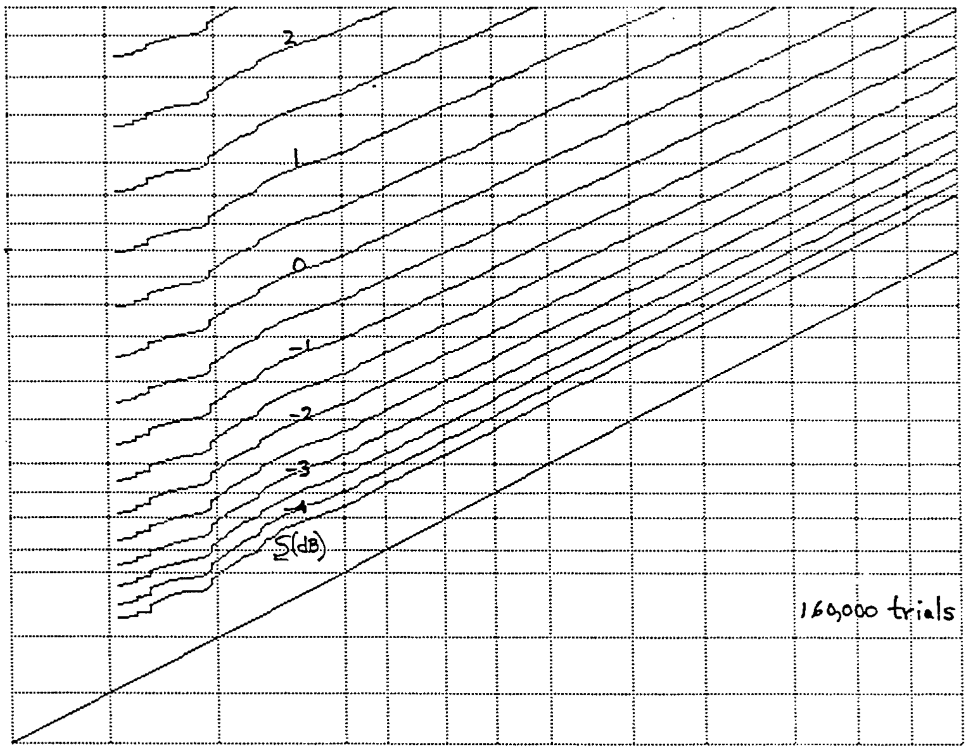

Figure F-13. ROC for M = 256, ν = 1, μ = 1, N = 1024

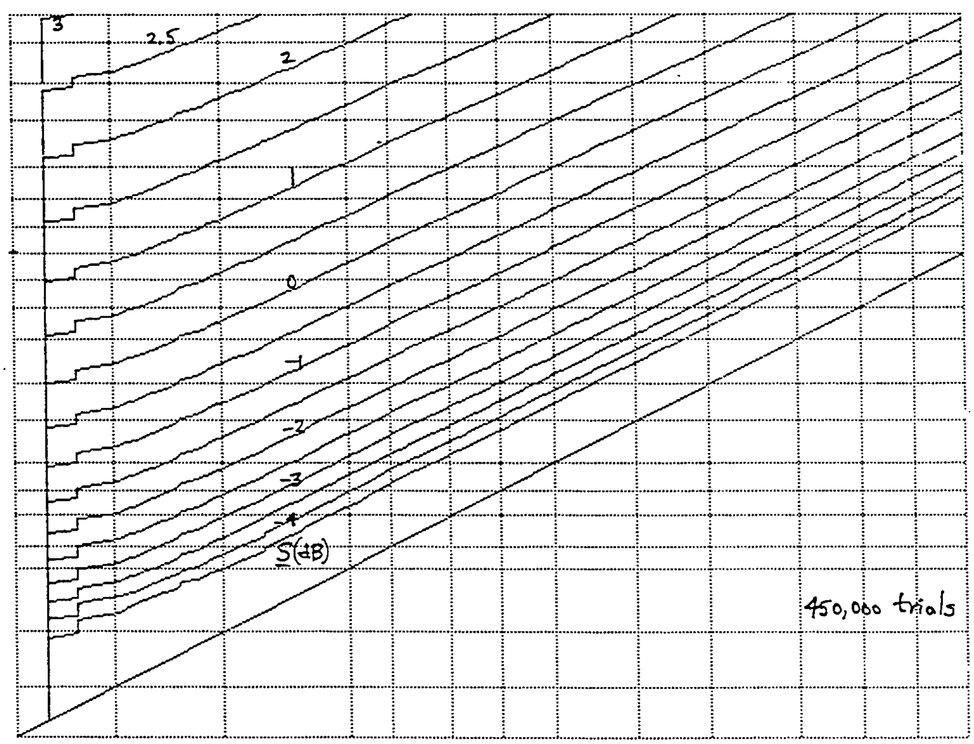

Figure F-14. ROC for M = 256, ν = 1.5, μ = 0.5, N = 1024

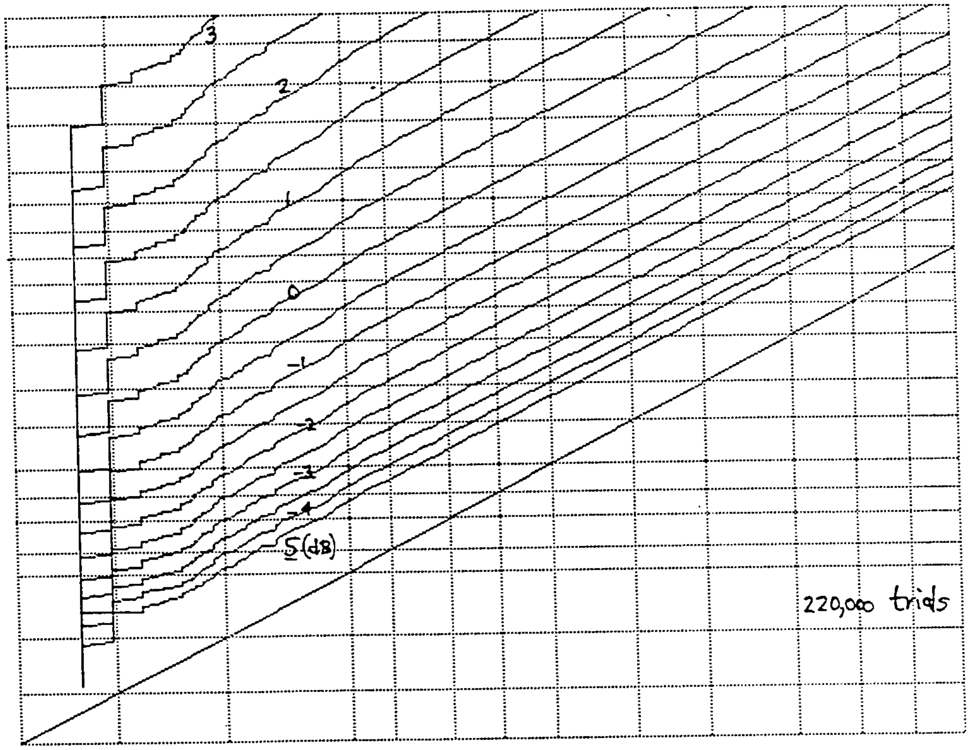

Figure F-15. ROC for M = 256, ν = 2, μ = 1.5, N = 1024

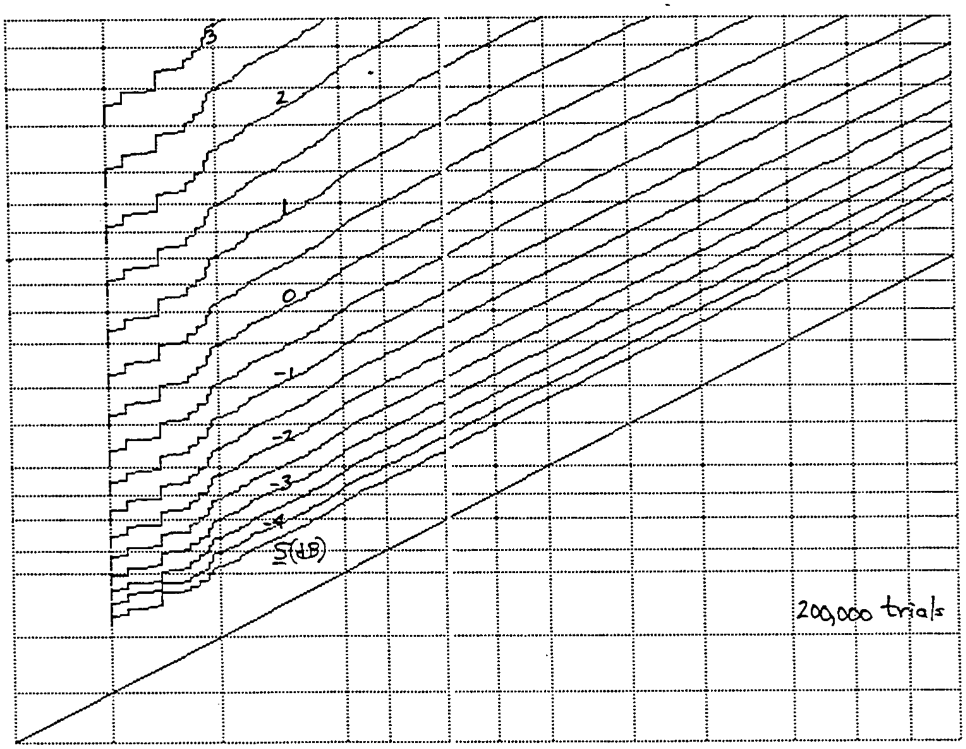

Figure F-16. ROC for M = 256, ν = 2.5, μ = 1, N = 1024

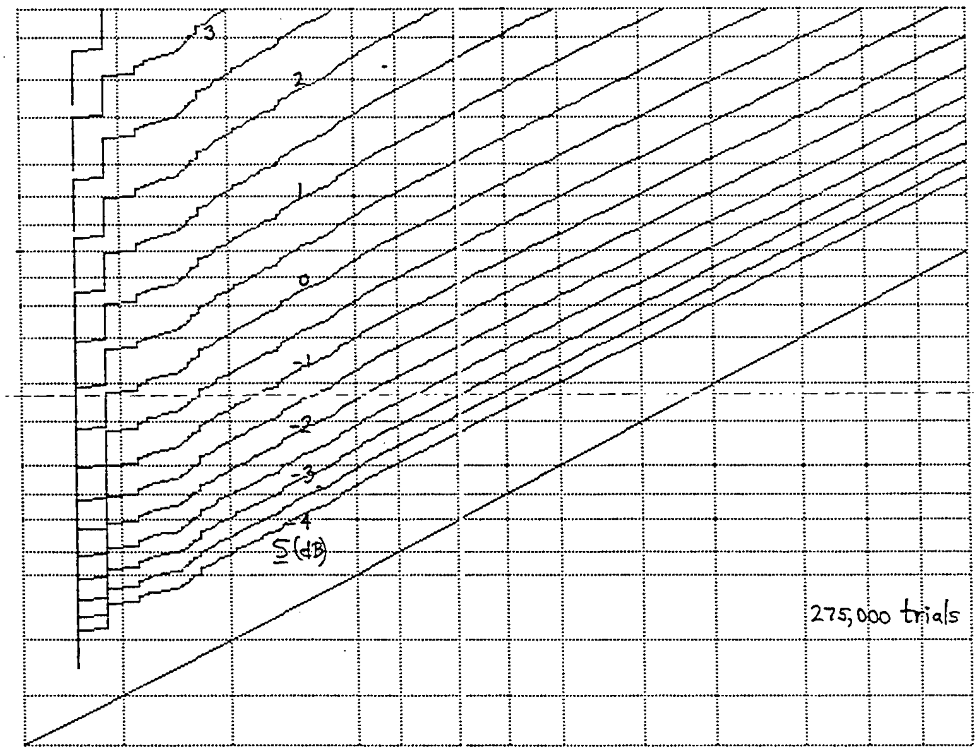

Figure F-17. ROC for M = 256, ν = 1.75, μ = 1.75, N = 1024

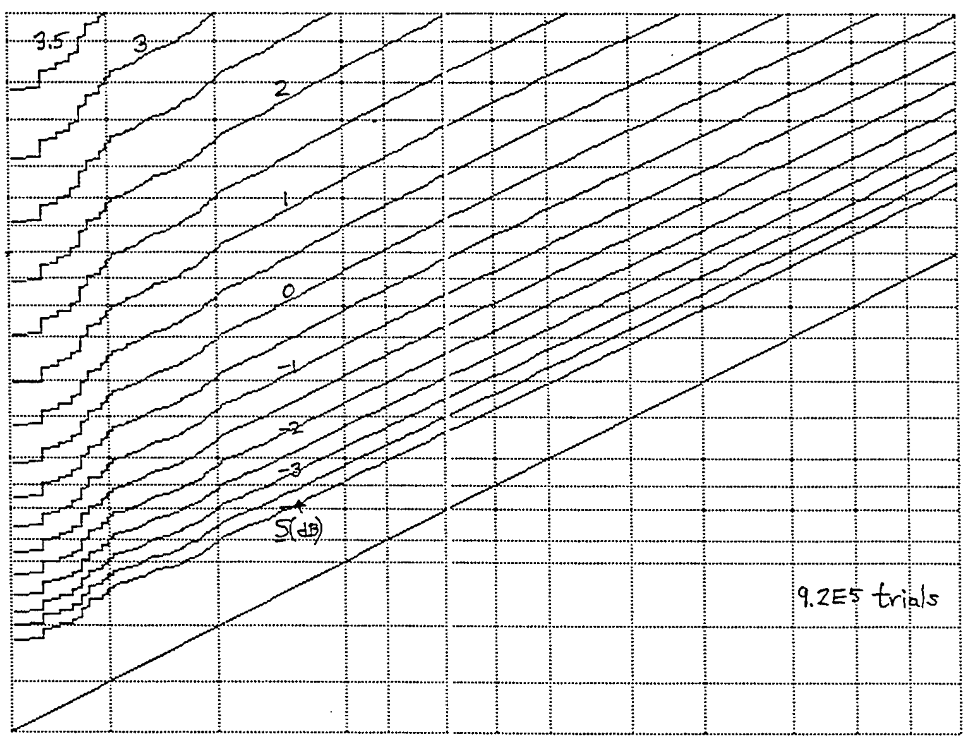

Figure F-18. ROC for M = 256, ν = 3, μ = 0, Ň = 1024

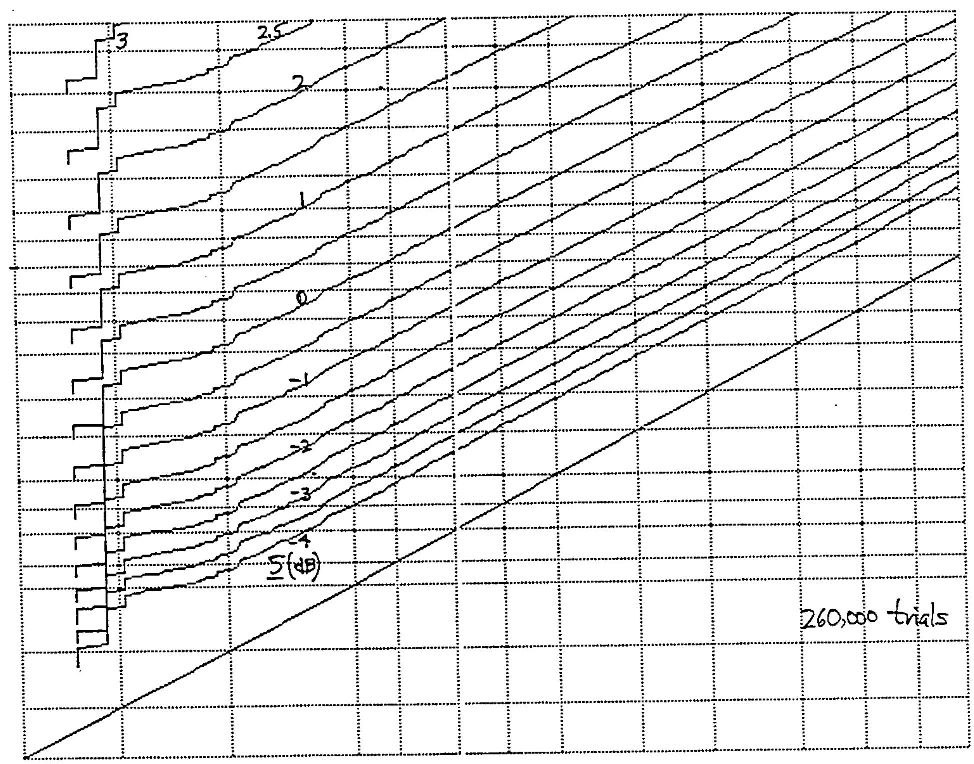

Figure F-19. ROC for $\underline{M}$ = 256, ν = 2.5, μ = 0, N = 1024

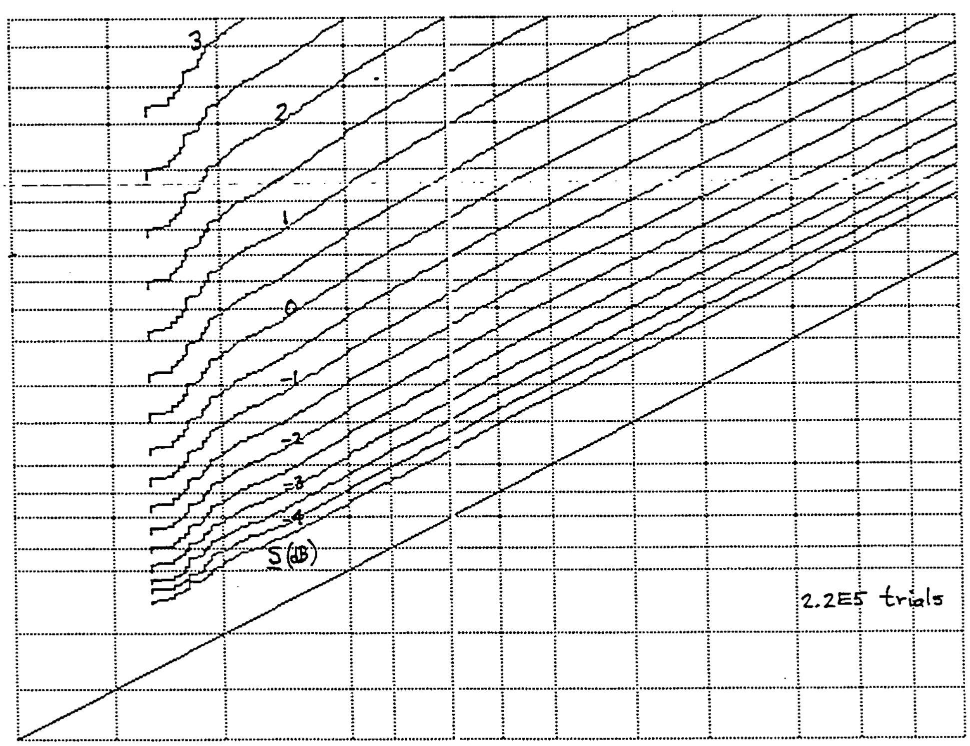

Figure F-20. ROC for $\underline{M}$ = 256, ν = 3, μ = 0.5, N = 1024

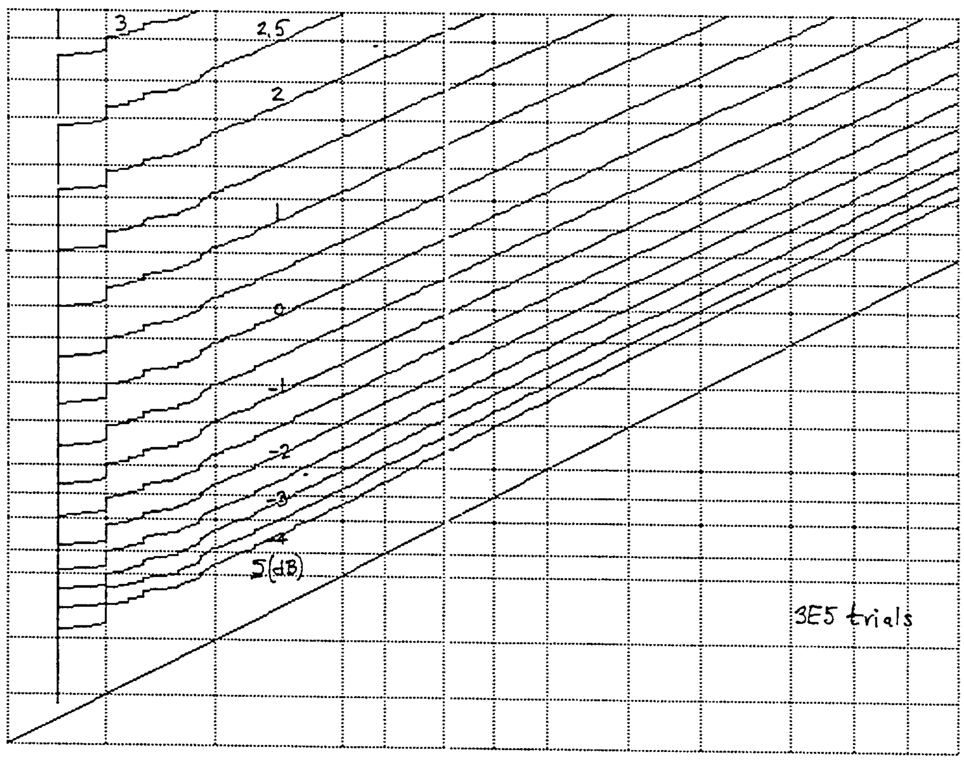

Figure F-21. ROC for $\underline{M}$ = 256, ν = 2, μ = 0, N = 1024

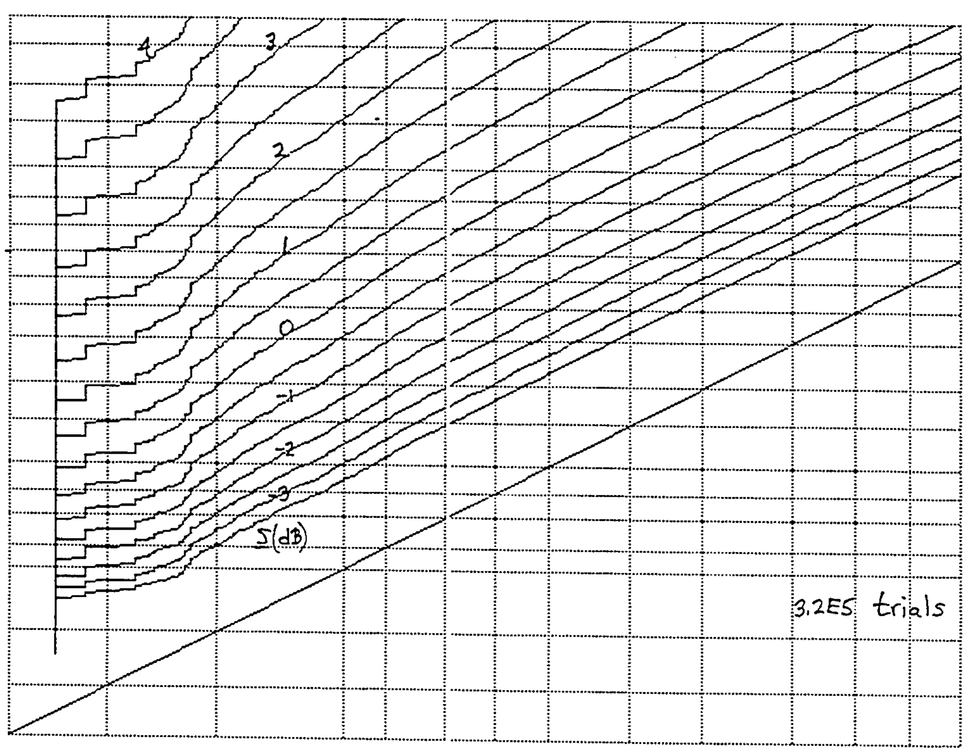

Figure F-22. ROC for $\underline{M}$ = 256, ν = 3, μ = 1, N = 1024

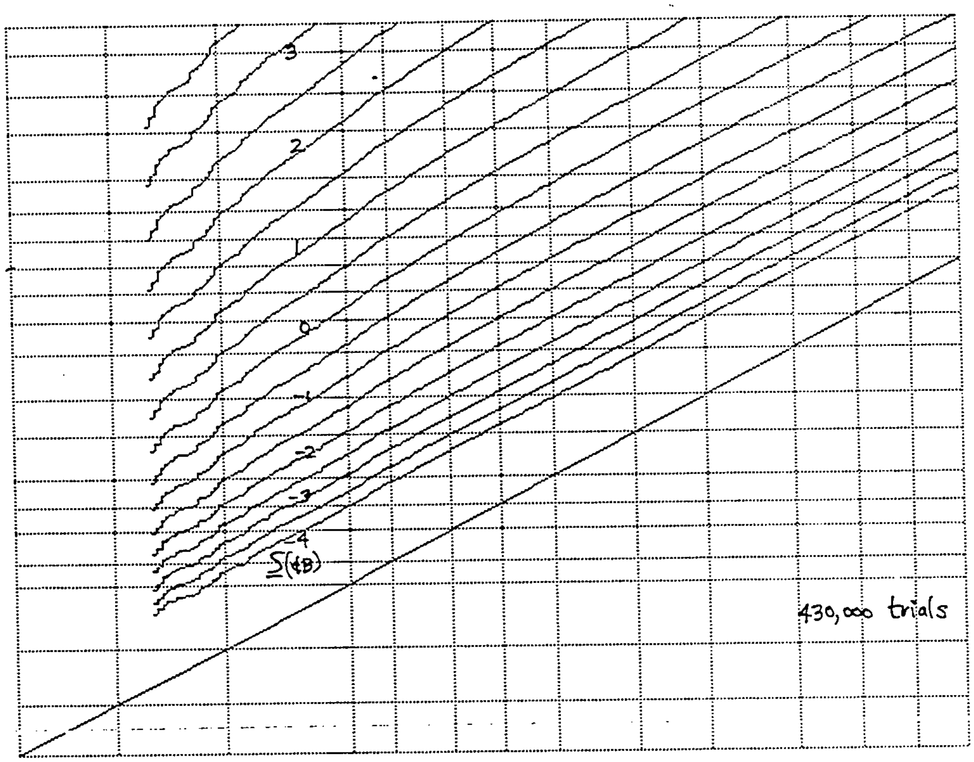

Figure F-23. ROC for M = 256, ν = 2.5, μ = 1.5, N = 1024

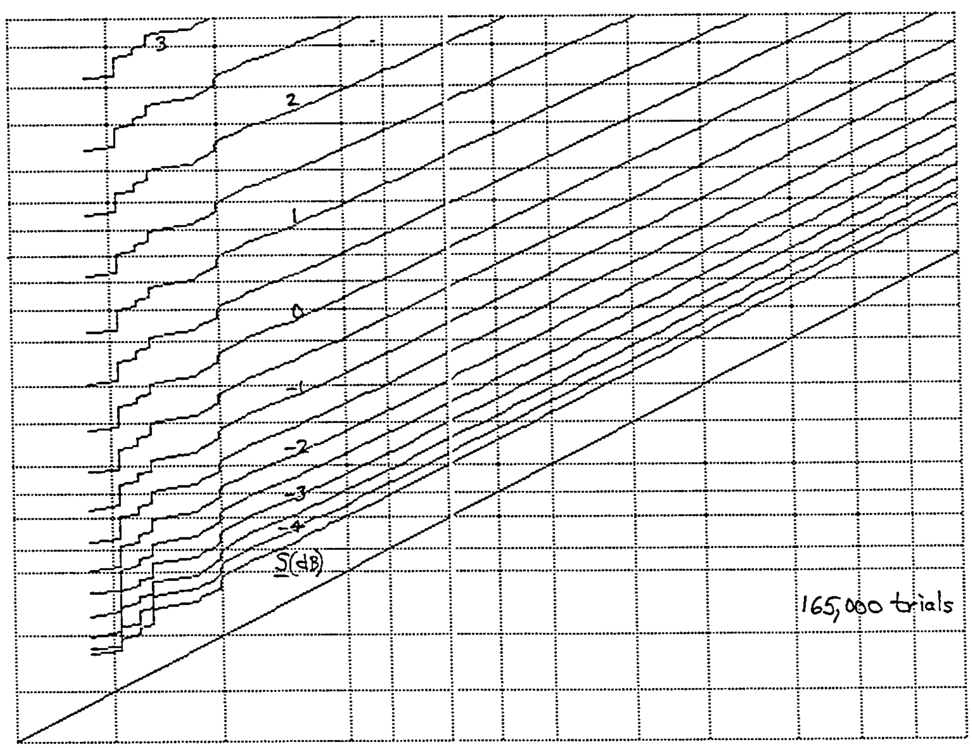

Figure F-24. ROC for M = 256, ν = 1, μ = 0.5, N = 1024

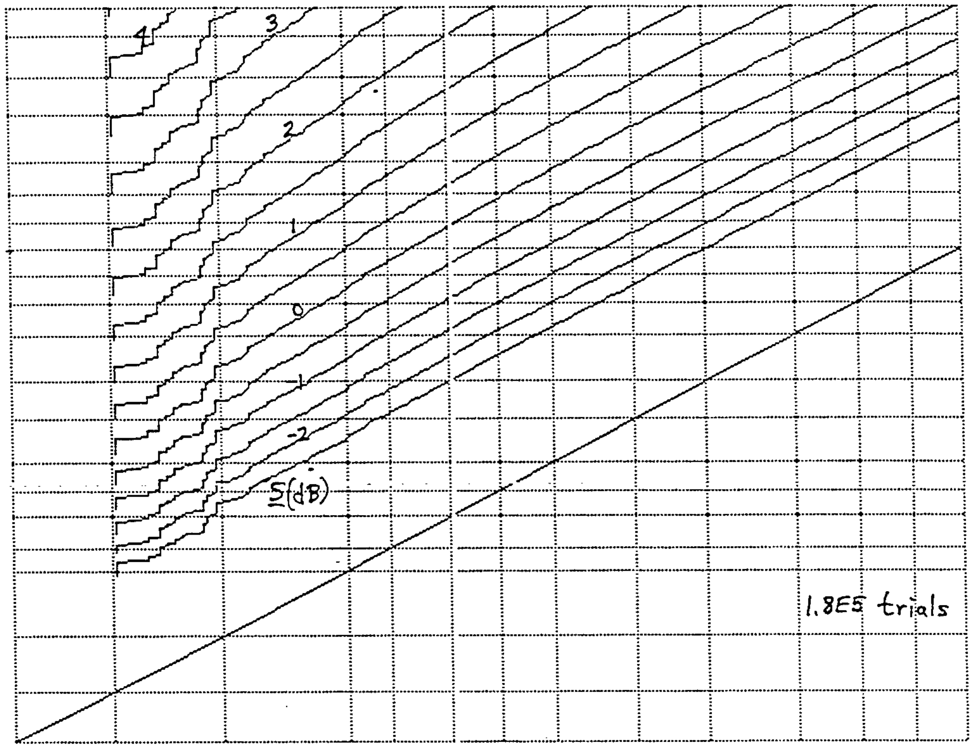

Figure F-25. ROC for M = 256, ν = 2, μ = 2, N = 1024

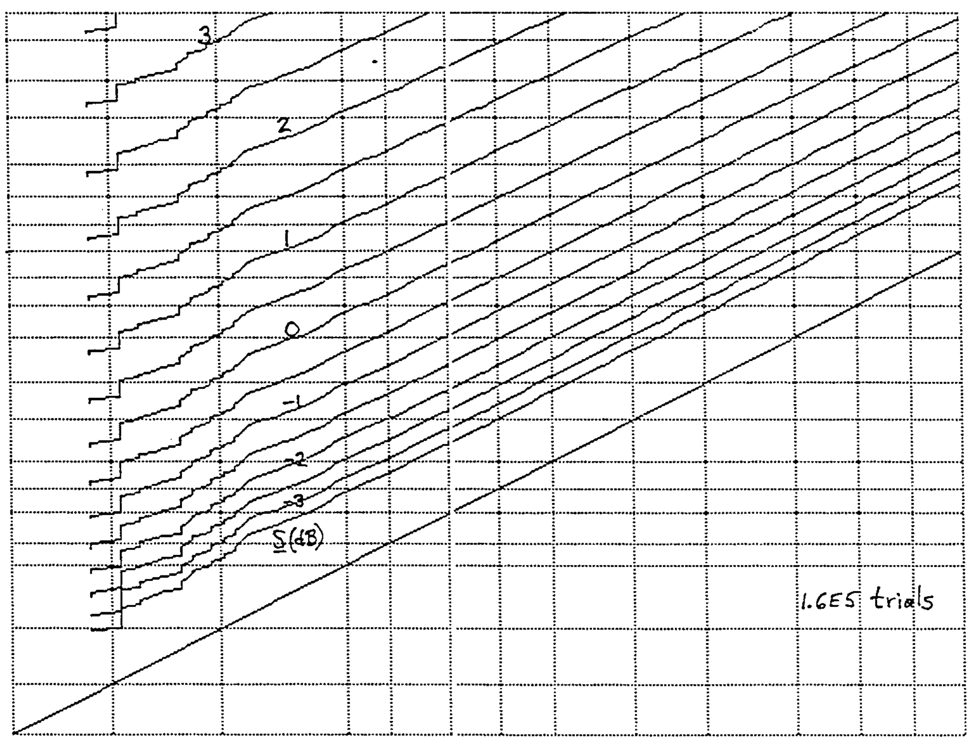

Figure F-26. ROC for M = 256, ν = 1, μ = 0.3, N = 1024

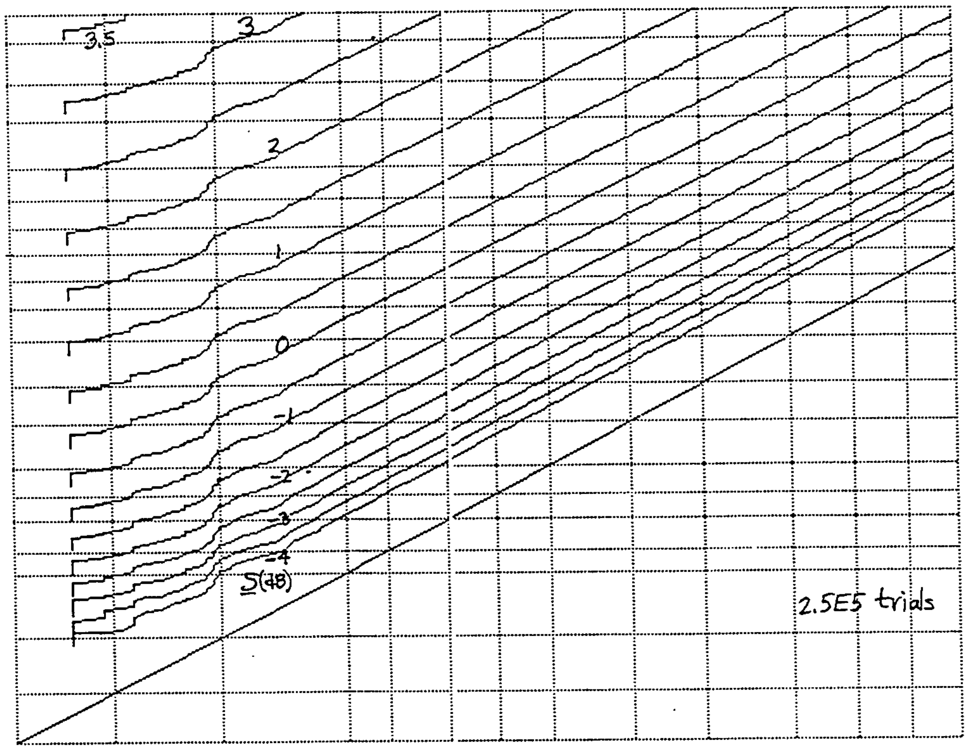

Figure F-27. ROC for $\underline{M}$ = 256, ν = 1.5, μ = 0, N = 1024

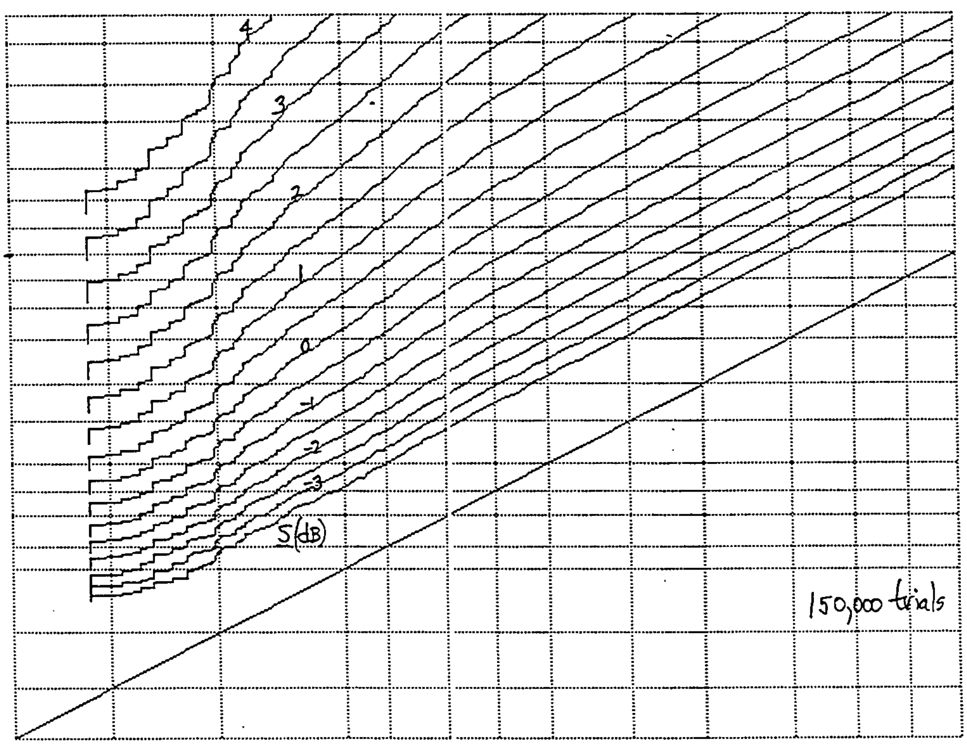

Figure F-28. ROC for $\underline{M}$ = 256, ν = 3, μ = 1.5, N = 1024

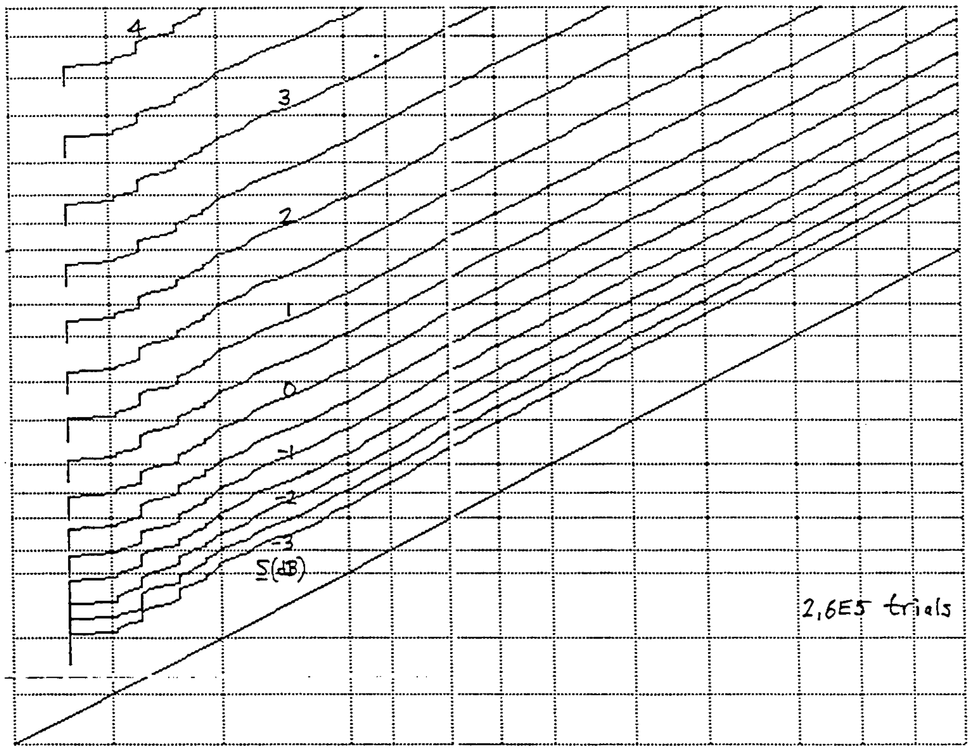

Figure F-29. ROC for M = 256, ν = 1, μ = 0, N = 1024

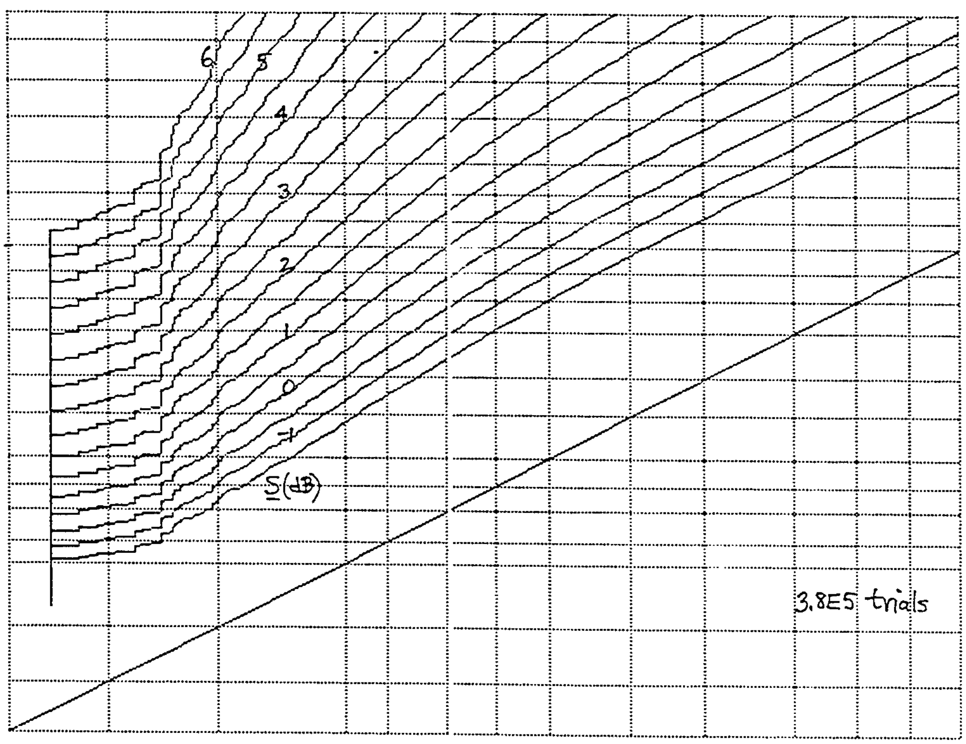

Figure F-30. ROC for M = 256, ν = 2.5, μ = 2.5, N = 1024

APPENDIX G — RECEIVER OPERATING CHARACTERISTICS FOR NORMALIZER (17)

The normalizer of interest here is

$$\frac{X(\nu,N)}{Z(\mu,L)} \gtrless v \;,$$

which is equation (17) from the main text. All the ROCs in this appendix are for $\underline{M}$ = 256, search size N = 1024, and reference size L = N = 1024. The values of power-law ν range over 1, 1.5, 2, 2.5, and 3, while averager parameter μ takes on the two values μ = 1 and μ = 0+. In this latter case, $Z(\mu,L)$ is the geometric mean of noise-only reference data $\{z_\ell\}$ (see equation (9)). The simulations were conducted for noise level $\underline{N}$ = 1; therefore, the signal level $\underline{S}$ (dB) labeled on each curve can be interpreted as the signal-to-noise ratio per bin in decibels. The number of independent trials of normalizer (17) that are utilized for each ROC are indicated in each case. The abscissa and ordinate labelings on figures G-2 to G-10 are identical to those indicated on figure G-1.

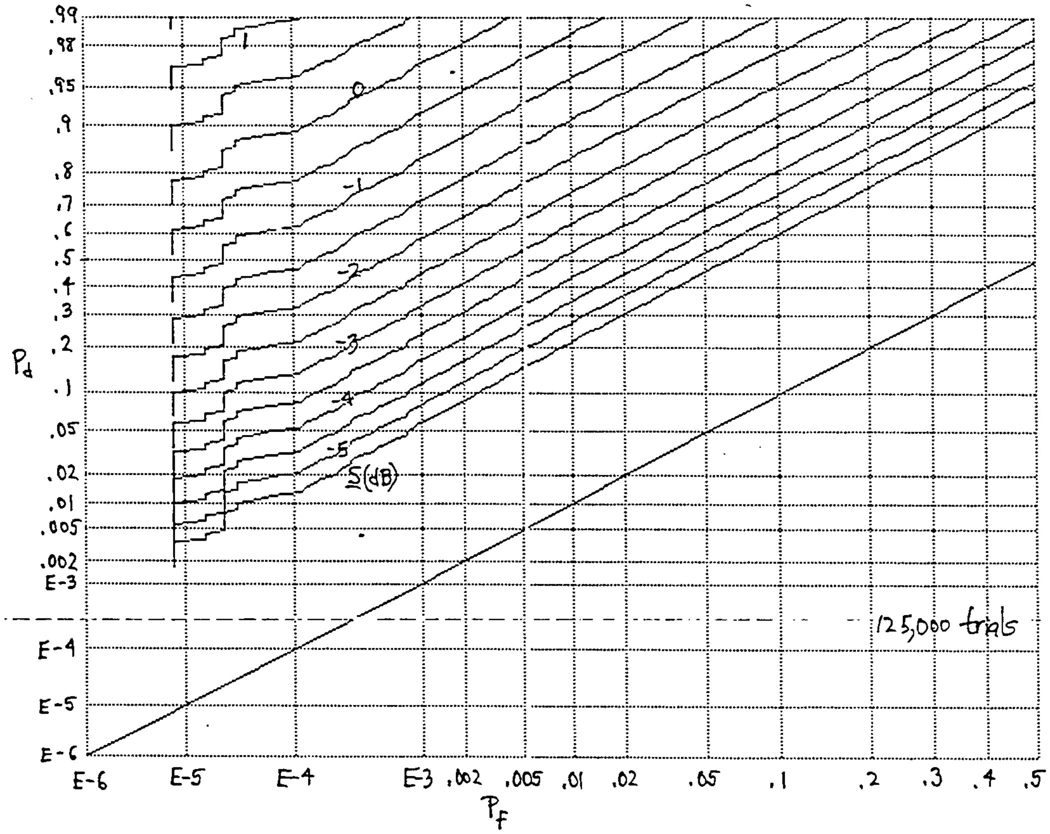

Figure G-1. ROC for $\underline{M}$ = 256, ν = 1, μ = 1, N = 1024, (17)

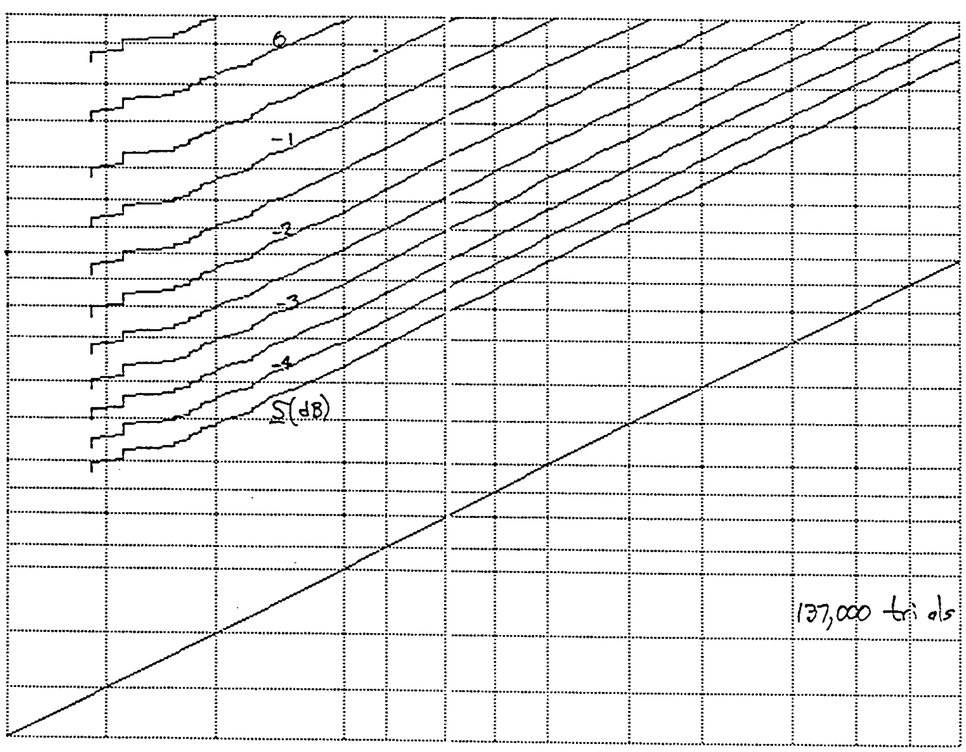

Figure G-2. ROC for $\underline{M}$ = 256, ν = 1.5, μ = 1, N = 1024, (17)

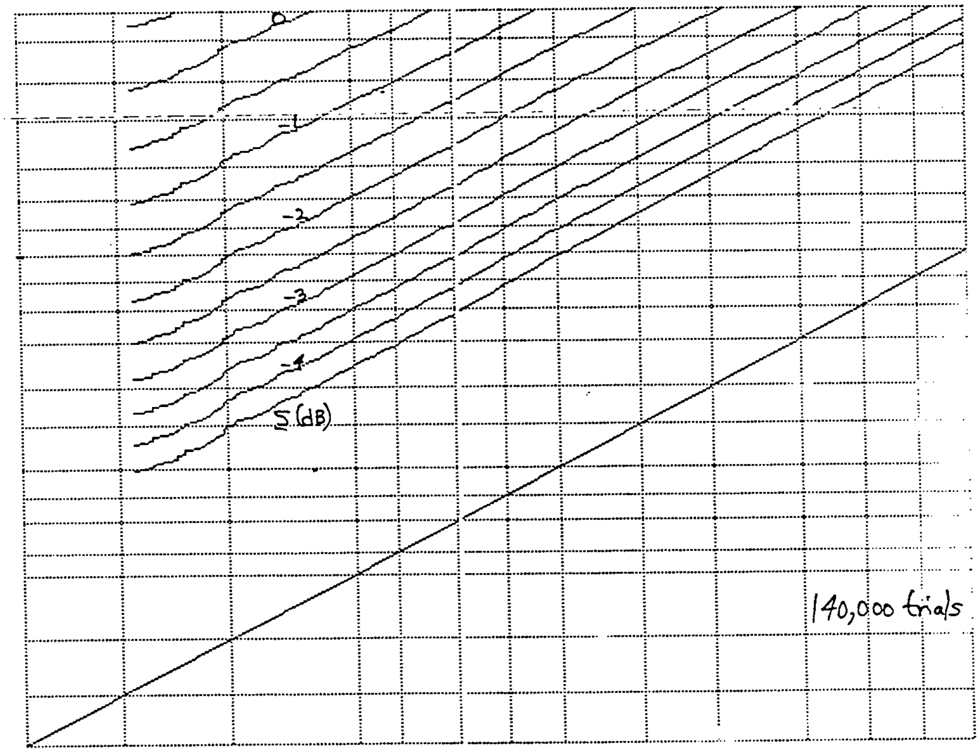

Figure G-3. ROC for $\underline{M}$ = 256, ν = 2, μ = 1, N = 1024, (17)

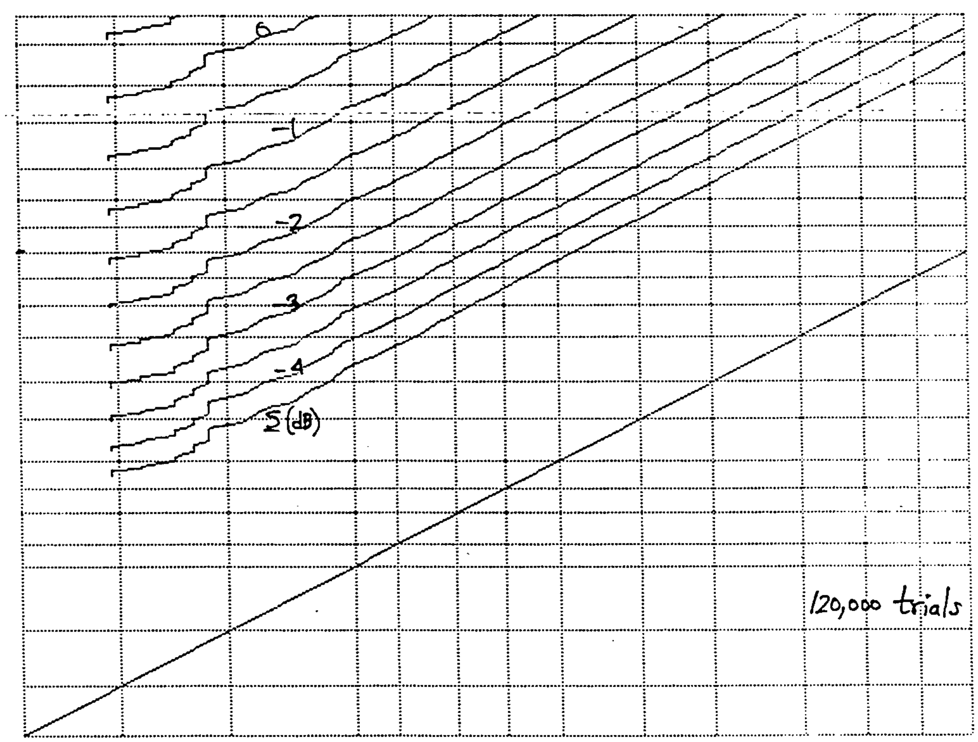

Figure G-4. ROC for $\underline{M}$ = 256, ν = 2.5, μ = 1, N = 1024, (17)

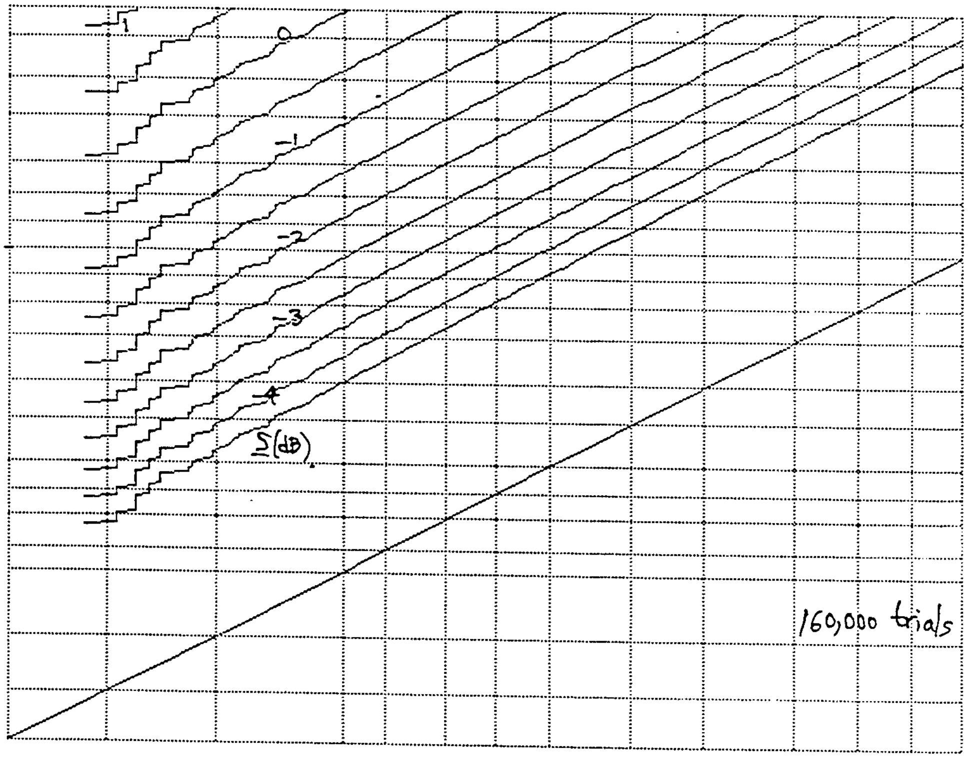

Figure G-5. ROC for $\underline{M}$ = 256, ν = 3, μ = 1, N = 1024, (17)

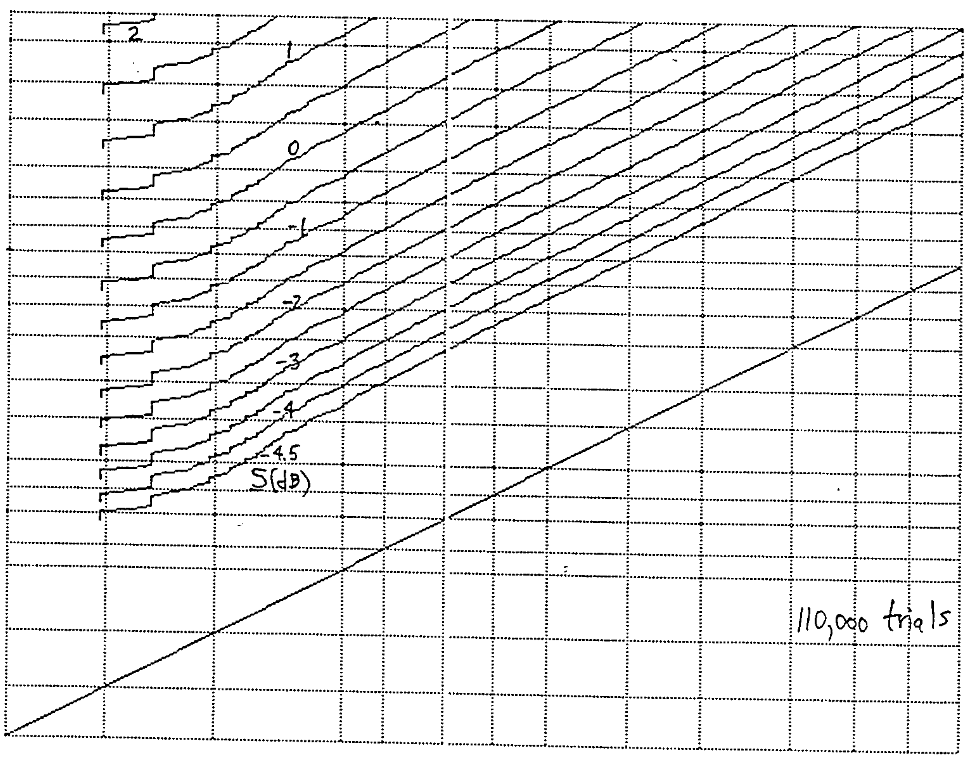

Figure G-6. ROC for $\underline{M}$ = 256, ν = 1, μ = 0, N = 1024, (17)

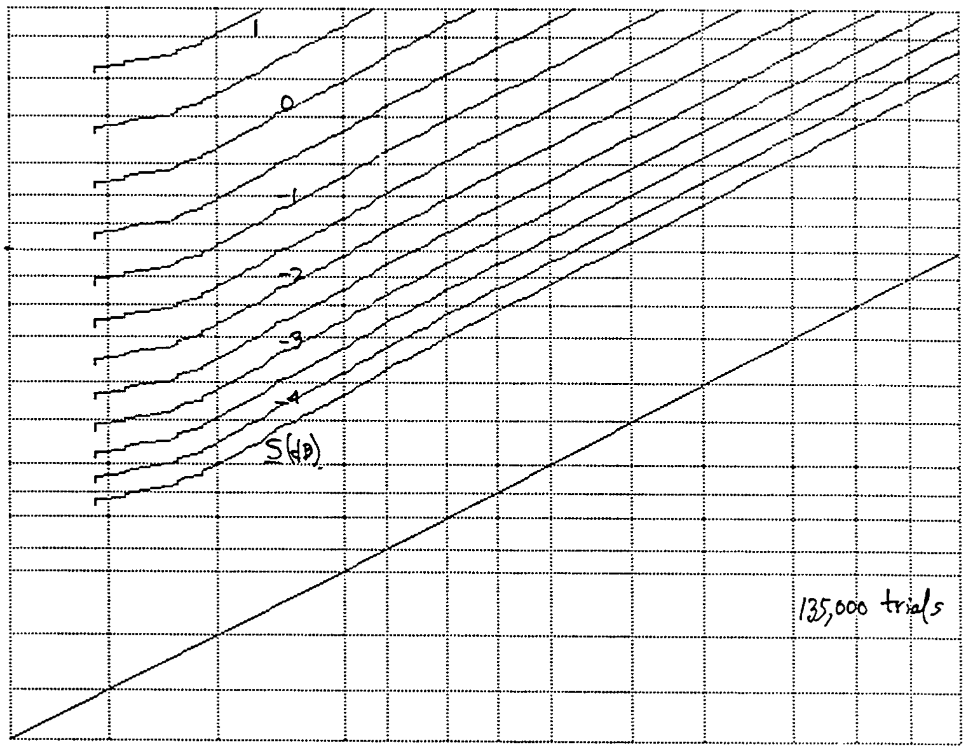

Figure G-7. ROC for $\underline{M}$ = 256, ν = 1.5, μ = 0, N = 1024, (17)

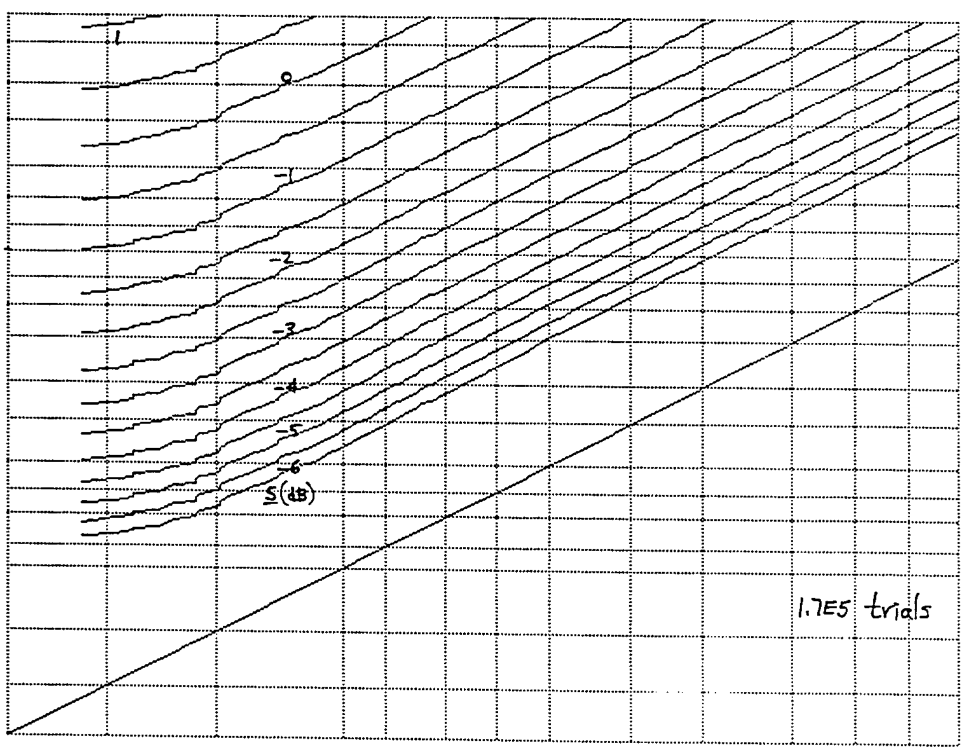

Figure G-8. ROC for $\underline{M}$ = 256, ν = 2, μ = 0, N = 1024, (17)

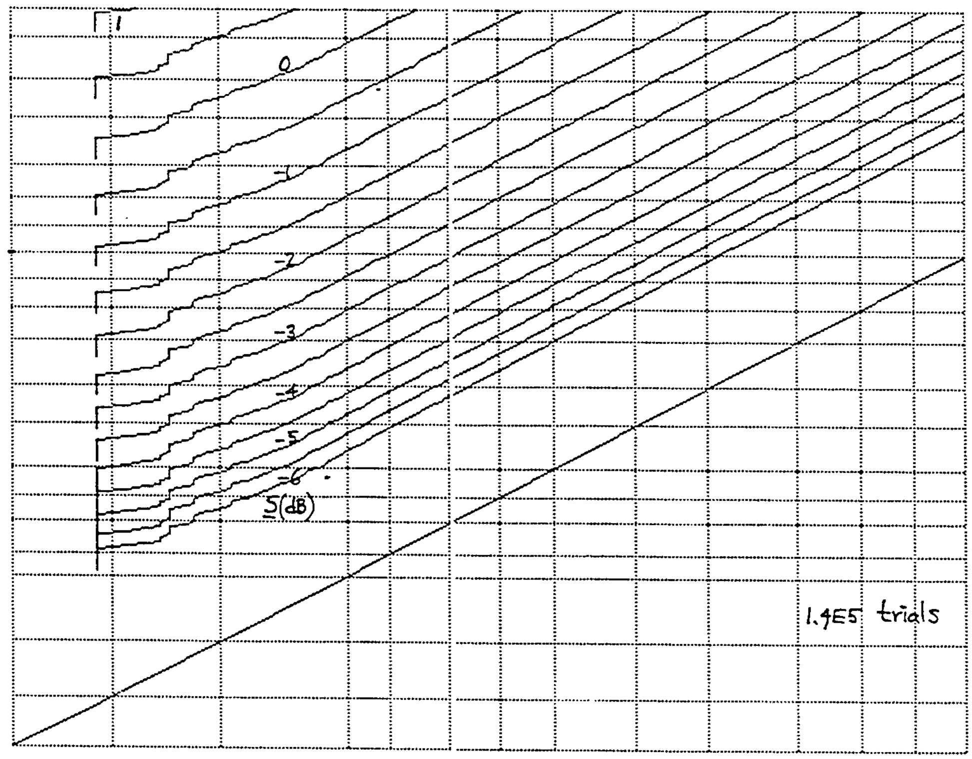

Figure G-9. ROC for $\underline{M}$ = 256, ν = 2.5, μ = 0, N = 1024, (17)

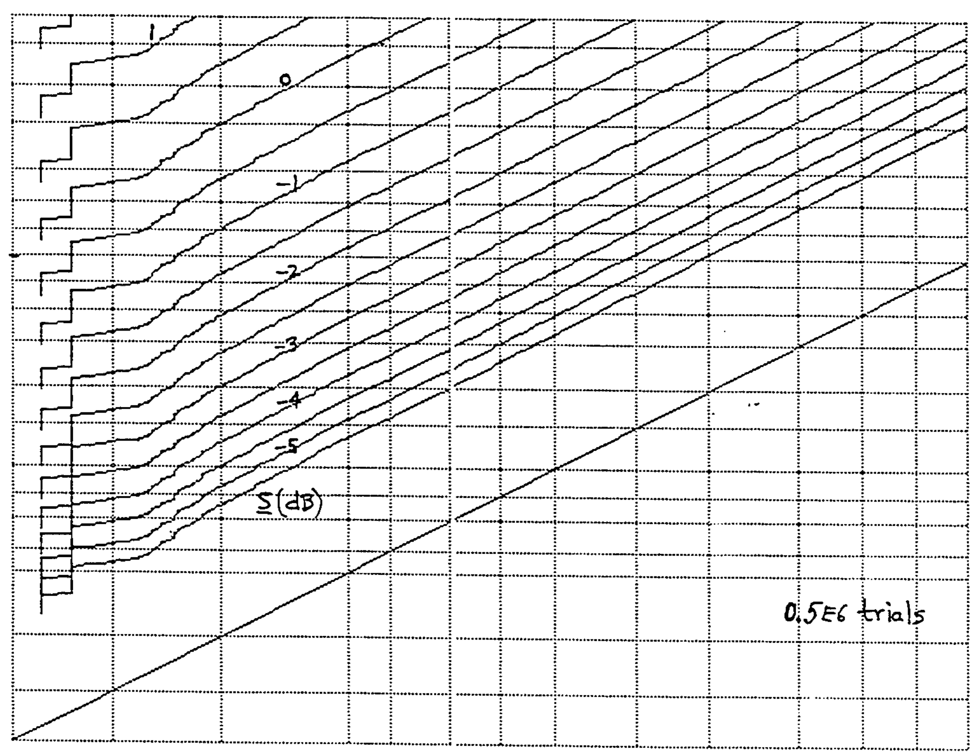

Figure G-10. ROC for $\underline{M}$ = 256, ν = 3, μ = 0, N = 1024, (17)

REFERENCES

1. A. H. Nuttall, "Detection Performance of a Modified Generalized Likelihood Ratio Processor for Random Signals of Unknown Location," NUWC-NPT Technical Report 10,539, Naval Undersea Warfare Center Detachment, New London, CT, 23 November 1993.

2. A. H. Nuttall, "Detection Performance of Generalized Likelihood Ratio Processors for Random Signals of Unknown Location, Structure, Extent, and Strength," NUWC-NPT Technical Report 10,739, Naval Undersea Warfare Center Division, Newport, RI, 25 August 1994.

3. A. H. Nuttall, "Detection Performance of Power-Law Processors for Random Signals of Unknown Location, Structure, Extent, and Strength," NUWC-NPT Technical Report 10,751, Naval Undersea Warfare Center Division, Newport, RI, 16 September 1994.

4. A. H. Nuttall, "Limiting Detection Performance for Random Signals of Unknown Location, Structure, Extent, and Strength," NUWC-NPT Technical Report 10,839, Naval Undersea Warfare Center Division, Newport, RI, 2 March 1995.

5. A. H. Nuttall, "Near-Optimum Detection Performance of Power-Law Processors for Random Signals of Unknown Locations, Structure, Extent, and Arbitrary Strengths," NUWC-NPT Technical Report 11,123, Naval Undersea Warfare Center Division, Newport, RI, 15 April 1996.

6. J. O. Ruanaidh and W. J. Fitzgerald, **Numerical Bayesian Methods Applied to Signal Processing**, Springer-Verlag Inc., New York, NY, 1996.

7. H. L. Van Trees, **Detection, Estimation, and Modulation Theory: Part I, Linear Modulation Theory**, John Wiley and Sons, Inc., New York, NY, 1968.

8. I. S. Gradshteyn and I. M. Ryzhik, **Table of Integrals, Series, and Products**, Academic Press, Inc., New York, NY, 1980.

9. A. H. Nuttall, "Accurate Efficient Evaluation of Cumulative or Exceedance Probability Distributions Directly From Characteristic Functions," NUSC Technical Report 7023, Naval Underwater Systems Center, New London, CT, 1 October 1983.

INITIAL DISTRIBUTION LIST

Addressee	No. of Copies
Center for Naval Analyses, VA	1
Coast Guard Academy, CT	
J. Wolcin	1
Defense Technical Information Center, VA	12
Griffiss Air Force Base, NY	
Documents Library	1
J. Michels	1
Hanscom Air Force Base, MA	
M. Rangaswamy	1
National Radio Astronomy Observatory, VA	
F. Schwab	1
National Security Agency, MD	
J. Maar	1
National Technical Information Service, VA	10
Naval Air Warfare Center, PA	
Commander	1
L. Allen	1
Naval Command Control and Ocean Surveillance Center, CA	
Commanding Officer	1
J. Alsup	1
W. Marsh	1
P. Nachtigall	1
C. Persons	1
C. Tran	1
Naval Environmental Prediction Research Facility, CA	1
Naval Intelligence Command, DC	1
Naval Oceanographic and Atmospheric Research Laboratory, CA	
M. Pastore	1
Naval Oceanographic and Atmospheric Research Laboratory, MS	
Commanding Officer	1
B. Adams	1
R. Fiddler	1
E. Franchi	1
R. Wagstaff	1
Naval Oceanographic Office, MS	1
Naval Personnel Research and Development Center, CA	1
Naval Postgraduate School, CA	
Superintendent	1
C. Therrien	1
Naval Research Laboratory, DC	
Commanding Officer	1
W. Gabriel	1
D. Steiger	1
E. Wald	1
N. Yen	1
Naval Research Laboratory, FL	
Superintendent	1
P. Ainsleigh	1

Addressee	No. of Copies
Naval Surface Warfare Center, FL	
Commanding Officer	1
E. Linsenmeyer	1
D. Skinner	1
Naval Surface Warfare Center, MD	
P. Prendergast	1
Naval Surface Warfare Center, VA	
J. Gray	1
Naval Surface Weapons Center, MD	
Officer in Charge	1
M. Strippling	1
Naval Surface Weapons Center, VA	
Commander	1
H. Crisp	1
D. Phillips	1
T. Ryczek	1
Naval Technical Intelligence Center, DC	
Commanding Officer	1
D. Rothenberger	1
Naval Undersea Warfare Center, FL	
Officer in Charge	1
R. Kennedy	1
Naval Weapons Center, CA	1
Office of the Chief of Naval Research, VA	
P. Abraham	1
R. Doolittle	1
N. Gerr	1
T. Goldsberry	1
D. Johnson	1
R. Tipper	1
J. Tague	1
A. van Tilborg	1
Space and Naval Warfare System Command, DC	
R. Holland	1
U.S. Air Force, Maxwell Air Force Base, AL	
Air University Library	1
U.S. Department of Commerce, CO	
A. Spaulding	1
U.S. Pacific Fleet, Pearl Harbor, HI	
J. Melillo	1
Vandenberg Air Force Base, CA	
R. Leonard	1

Addressee	No. of Copies
Brown University, RI	
Documents Library	1
Catholic University, DC	
J. McCoy	1
Drexel University, PA	
S. Kesler	1
Duke University, NC	
J. Krolik	1
Harvard University, MA	
Gordon McKay Library	1
Kansas State University, KS	
B. Harms	1
Lawrence Livermore National Laboratory, CA	
Director	1
L. Ng	1
Los Alamos National Laboratory, NM	1
Marine Biological Laboratory, MA	1
Massachusetts Institute of Technology, MA	
Barker Engineering Library	1
Northeastern University, MA	
C. Nikias	1
Pennsyvania State University, PA	
Director	1
R. Hettche	1
E. Liszka	1
F. Symons	1
Princeton University, NJ	
S. Schwartz	1
Rutgers University, NJ	
S. Orfanidis	1
San Diego State University, CA	
F. Harris	1
Sandia National Laboratory, NM	
Director	1
J. Claasen	1
Scripps Institution of Oceonography, CA	1
Southeastern Massachusetts University, MA	
C. Chen	1
Syracuse University, NY	
D. Weiner	1
United Engineering Center, NY	
Engineering Societies Library	1
University of Colorado, CO	
L. Scharf	1

Addressee	No. of Copies
University of Connecticut, CT	
Wilbur Cross Library	1
C. Knapp	1
P. Willett	1
University of Florida, FL	
D. Childers	1
University of Hartford	
Science and Engineering Library	1
University of Illinois, IL	
D. Jones	1
A. Nehorai	1
University of Michigan, MI	
Communications and Signal Processing Laboratory	1
W. Williams	1
University of Minnesota, MN	
M. Kaveh	1
University of Rhode Island, RI	
Library	1
G. Boudreaux-Bartels	1
S. Kay	1
D. Tufts	1
University of Rochester, NY	
E. Titlebaum	1
University of Southern California, CA	
W. Lindsey	1
A. Polydoros	1
University of Texas, TX	1
University of Washington, WA	
Applied Physics Laboratory	1
D. Lytle	1
J. Ritcey	1
R. Spindel	1
Villanova University, PA	
M. Amin	1
Woods Hole Oceanographic Institution, MA	
Director	1
K. Fristrup	1
T. Stanton	1
Yale University, CT	
Kline Science Library	1
P. Schultheiss	1

Addressee	No. of Copies
Analysis and Technology, CT	
Library	1
Analysis and Technology, VA	
D. Clark	1
Bell Communications Research, NJ	
D. Sunday	1
Berkeley Research, CA	
S. McDonald	1
Bolt, Beranek, and Newman, CT	
P. Cable	1
Bolt, Beranek, and Newman, MA	
H. Gish	1
EDO Corporation, NY	
M. Blanchard	1
E G & G, VA	
D. Frohman	1
Engineering Technology Center	
D. Lerro	1
General Electric, MA	
R. Race	1
General Electric, NJ	
H. Urkowitz	1
Harris Scientific Services, NY	
B. Harris	1
Hughes Aircraft, CA	
T. Posch	1
Kildare Corporation, CT	
R. Mellen	1
Lincom Corporation, MA	
T. Schonhoff	1
Magnavox Electronics Systems, IN	
R. Kenefic	1
MITRE Corporation, CT	
S. Pawlukiewicz	1
Nichols Research, MA	
T. Marzetta	1
Orincon Corporation, CA	
J. Young	1
Orincon Corporation, VA	
H. Cox	1
Prometheus, RI	
M. Barrett	1
J. Byrnes	1
Raytheon, RI	
R. Conner	1
S. Reese	1

Addressee	No. of Copies
Rockwell International, CA	
D. Elliott	1
Schlumberger-Doll Research, CT	
R. Shenoy	1
Science Applications International Corporation, CA	
C. Katz	1
Science Applications International Corporation, CT	
F. DiNapoli	1
Science Applications International Corporation, VA	
P. Mikhalevsky	1
Sperry Corporation, NY	1
Toyon Research, CA	
M. Van Blaricum	1
Tracor, TX	
T. Leih	1
B. Jones	1
K. Scarbrough	1
Tracor, CA	
M. van Holliday	1
TRW, VA	
R. Prager	1
G. Maher	1
Westinghouse Electric, MA	
R. Kennedy	1
Westinghouse Electric, MD	
H. Newman	1
Westinghouse Electric, MD	
R. Park	1